The Solutions of Second Order Nonlinear Two Point Boundary Value Problems: Generalized Shifted Legendre Polynomials, Homotopy Continuation Method

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Abstract—In this paper, generalized shifted Legendre polynomial approximation on a given arbitrary interval has been designed to find an approximate solution of a given second order nonlinear two-point boundary value problems of ordinary differential equations. Here an approach using Tau method based on Legendre operational matrix of differentiation has been addressed to generate the nonlinear systems of algebraic equations. The unknown Legendre coefficients of these nonlinear systems are the solutions of the system and they have been solved by continuation method. These unknown Legendre coefficients are then used to write the approximate solutions to the second order nonlinear two-point boundary value problems. The validity and efficiency of the method has also been illustrated with numerical examples and graphs assisted by MATLAB.

Index Terms—Boundary Value Problems (BVPs), Generalized Shifted Legendre Polynomials, Homotopy Continuation Method, Legendre Operational Matrix of Differentiation, Nonlinear Ordinary Differential Equations.

I. INTRODUCTION

According to [1], second order two-point boundary value problems of ordinary differential equations are equations of the form

\[ y'' = f(x, y, y') , a \leq x \leq b, \]

With the boundary conditions on the solution prescribed by

\[ y(a) = \alpha , y(b) = \beta, \]

for some constants \( \alpha \) and \( \beta \). Second order differential equations with various types of boundary conditions are among many of linear and nonlinear problems occurring in science and engineering which can be solved either analytically or numerically [3]. Second order two-point boundary value problems are encountered in many engineering fields including optimal control, beam deflections, heat flow, and various dynamical systems [4].

In the literature of numerical analysis for solving a two point second order boundary value problems (BVPs) of differential equations, many authors have attempted to obtain higher accuracy rapidly by using numerous methods. Among various numerical techniques, finite difference method has been widely used but it takes more computational costs to get high accuracy [3]. In [4] the author has applied a cubic B-spline method to find the solutions of both linear and nonlinear second order two point BVPs of ordinary differential equations. The authors in [6] have used an extended cubic B-spline method for solving linear two point BVPs. In [7] the authors found the solution of two-point boundary value problems using quartic B-spline method. But these B-spline methods require dividing the interval \([a,b]\) into \(n\) subintervals and the construction of cubic or extended cubic B-splines in each subinterval. In addition while using these methods; we need to solve \(n+1\) systems of nonlinear equations to arrive at better accuracy. Therefore, like finite difference method the methods are also computationally too cost. In [8] the authors have developed Galerkin method to approximate the solution of second order Neumann and Cauchy linear boundary value problems. The authors in [9] derived a new difference scheme for solving linear and nonlinear second order two-point boundary value problems by using the quartic spline interpolation and Taylor expansion. The author in [10] found the approximate solutions of second order linear and nonlinear boundary value problems using shifted Legendre polynomials on specific interval \([0,1]\).

Finding the solution of ordinary differential equations numerically is not only concerned with getting better accuracy. It is also concerned with saving computational time and effort. This paper is therefore aimed at finding the solutions of general second order nonlinear two-point boundary value problems of ordinary differential equations of the form:

\[ y''(x) + p(x)y'(x) + f(x, y) = g(x) , a \leq x \leq b \]  \hspace{1cm} (1)

Subject to the boundary conditions:

\[ y(a) = \alpha , y(b) = \beta \]  \hspace{1cm} (2)

using generalized shifted Legendre polynomial approximation on a given arbitrary interval combined with homotopy continuation method.

This study is therefore, a contribution towards finding the solutions of general second order nonlinear two-point boundary value problems of ordinary differential equations, using generalized shifted Legendre polynomial approximation on a given arbitrary interval combined with Homotopy Continuation method. Even though many authors have attempted to obtain higher accuracy rapidly by using
numerous methods like finite difference method and spline methods such as cubic B-spline method, an extended cubic B-spline method and quartic B-spline method, each of these methods are computationally too cost and slow when compared to the present method. Moreover, the present paper finds the solution on an arbitrary interval \([a, b]\).

Thus, the purpose of this study was to find the solutions of second order nonlinear two-point boundary value problems of ordinary differential equations, by using generalized shifted Legendre polynomial approximation on some arbitrary interval combined with Homotopy Continuation method.

The advantage of this method over the other methods is therefore
- It is computationally economical,
- It needs less computational time and effort,
- It has better accuracy and
- It enables us to solve boundary value problems imposed on any arbitrary interval.

The remainder of this paper has been organized in the following order and procedures.

- In section II, generalized shifted Legendre polynomial on some arbitrary interval \([a, b]\) for \(a < b\) has been derived.
- In section III, the application of this method combined with Homotopy Continuation method has been discussed in brief.
- In section IV, this method has been illustrated with numerical examples and graphs.
- The paper has been concluded and recommended in section V.

II. GENERALIZED SHIFTED LEGENDRE POLYNOMIALS FOR ARBITRARY INTERVAL

Legendre Polynomials are defined on the interval \([-1, 1]\) and can be determined with the aids of the following recurrence formulae [5].

\[
L_0(t) = 1, \quad L_1(t) = t \\
L_{r+1}(t) = \frac{2r+1}{r+1} (t) L_r(t) - \frac{r}{r+1} L_{r-1}(t); \quad r = 1, 2, 3, \ldots \tag{3}
\]

In order to use this polynomials over arbitrary interval \([a, b]\) with \(a < b\), we define the generalized shifted Legendre polynomial by introducing \(t = \frac{2x-(a+b)}{b-a}\). Let the generalized shifted Legendre polynomial \(L_r(2x-1)\) be denoted by \(P_r(x)\). Then \(P_r(x)\) can also be obtained as :

\[
P_0(x) = 1, \quad P_1(x) = \frac{2x-(a+b)}{b-a} \\
P_{r+1}(x) = \frac{2r+1}{r+1} \left(\frac{2x-(a+b)}{b-a}\right) P_r(x) - \frac{r}{r+1} P_{r-1}(x); \quad r = 1, 2, 3, \ldots \tag{4}
\]

The analytical form of the generalized shifted Legendre polynomial \(P_r(x)\) of degree \(r\) is given by:

\[
P_r(x) = \sum_{k=0}^{r} (-1)^{r+k} \frac{(r+k)!}{(r-k)! \cdot k!} x^k \tag{5}
\]

**NB:** \(P_r(0) = (-1)^r\) and \(P_r(1) = 1\)

The orthogonality condition for these the generalized shifted Legendre polynomials are:

\[
\int_a^b P_r(x) P_s(x) \, dx = \begin{cases} 
\frac{1}{2r+1} & \text{for } r = s \\
0 & \text{for } r \neq s
\end{cases} \tag{6}
\]

Any function \(y(x) \in L^2[a, b]\) can be approximated in terms of \(P_r(x)\) by:

\[
\tilde{y}(x) = \sum_{r=0}^{\infty} c_r P_r(x) \tag{7}
\]

Where the coefficients \(c_r\) are given by

\[
c_r = (2r + 1) \int_a^b y(x) P_r(x) \, dx ; \quad r = 1, 2, 3, \ldots \tag{8}
\]

By considering only the first \(m + 1\) terms of the series (7) we get;

\[
\tilde{y}(x) = \sum_{r=0}^{m} c_r P_r(x) = C^T \varphi(x) \tag{9}
\]

Where \(C^T = [c_0, c_1, \ldots, c_m]\) is the shifted Legendre coefficient and

\[
\varphi(x) = [p_0(x), p_1(x), \ldots, p_m(x)]^T
\]

is the shifted Legendre vector.

The derivative of the vector \(\varphi(x)\) can be expressed as:

\[
\frac{d\varphi(x)}{dx} = D^{(1)} \varphi(x) \tag{10}
\]

Where \(D^{(1)}\) is \((m+1) \times (m+1)\) operational matrix of derivative which is given by

\[
D^{(1)}(i,j) = \begin{cases} 
(2j+1) & \text{for } j = i - k; \\
0 & \text{ otherwise}
\end{cases}
\]

\[
\begin{cases} 
k = 1, 3, \ldots, m & \text{if } m \text{ is odd} \\
k = 1, 3, \ldots, m-1 & \text{if } m \text{ is even}
\end{cases}
\]

For example for \(m\) even we have;

\[
D^{(1)} = 2 \begin{pmatrix} 
0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
1 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 3 & 0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
1 & 0 & 5 & 0 & \ldots & 0 & 0 \\
0 & 3 & 0 & 7 & \ldots & 0 & \omega
\end{pmatrix}
\]

where

\[
\tau = 2m - 3, \omega = 2m - 1 \tag{11}
\]
From (10) it can be generalized for any \( n \in N \) as:
\[
\frac{d^n \phi(x)}{dx^n} = (D^{(1)})^n \phi(x) = D^{(n)} \phi(x), \quad n = 1, 2, 3, ...
\] (12)

Where, \((D^{(1)})^n\) denotes matrix powers.

### III. APPLICATIONS OF THE GENERALIZED SHIFTED LEGENDRE POLYNOMIALS COMBINED WITH HOMOTOPY CONTINUATION METHOD

Consider the general second order nonlinear two-point boundary value problem of ordinary differential equation
\[
y''(x) + p(x)y'(x) + f(x, y) = g(x), \quad a \leq x \leq b
\]
Subject to the boundary conditions:
\[
y(a) = \alpha, \quad y(b) = \beta
\]
As in [2], let us approximate \( y(x), p(x), f(x, y) \) and \( g(x) \) by the generalized shifted Legendre polynomials as
\[
y(x) = \sum_{r=0}^n c_r \varphi_r(x) = C^T \varphi(x)
\] (13)
\[
p(x) = \sum_{r=0}^n p_r \varphi_r(x) = P^T \varphi(x)
\] (14)
\[
f(x, y) = f(x, C^T \varphi(x))
\] (15)
\[
g(x) = \sum_{r=0}^n g_r \varphi_r(x) = G^T \varphi(x)
\] (16)

Where the unknowns are
\[
C = [c_0, c_1, \ldots, c_m]^T
\]

Using Legendre operational matrix of differentiation, (1) can be written as
\[
C^T D^2 \varphi(x) + P^T D^2 \varphi(x) + f(x, C^T \varphi(x)) = G^T \varphi(x)
\] (17)

The residual \( R_m(x) \) for (17) can be written as
\[
R_m(x) = C^T D^2 \varphi(x) + P^T D^2 \varphi(x) + f(x, C^T \varphi(x)) - G^T \varphi(x)
\] (18)

Applying typical Tau method, which is used in the sense of particular form of the Petrov-Galerkin method as cited in [2], [5], equation (17) can be transformed into \( m+1 \) nonlinear systems of equations by applying
\[
\langle R_m(x), P_r(x) \rangle = \int_a^b R_m(x) P_r(x) \, dx = 0;
\]
\[
r = 0, 1, 2, \ldots, m - 2
\] (19)

The boundary conditions are given by
\[
y(a) = C^T \varphi(a) = d_a y(b) = C^T \varphi(b) = d_b
\] (20)

Equations (19) and (20) generate \( m+1 \) nonlinear systems of algebraic equations whose coefficients are the unknowns in vectors \( C \). A homotopy continuation method is then applied to find these unknowns which are used to write the approximate solution \( \tilde{y}(x) \).

#### A. Homotopy

It has been briefly discussed in [1], that, Homotopy, or continuation, methods for nonlinear systems embed the problem to be solved within a collection of problems. Specifically, to solve a problem of the form \( F(c) = 0 \), which has the unknown solution \( c^* \), we consider a family of problems described using a parameter \( \lambda \) that assumes values in \([0,1]\). A problem with a known solution \( c(0) \) corresponds to the situation when \( \lambda = 0 \), and the problem with the unknown solution \( c(1) = c^* \) corresponds to \( \lambda = 1 \).

Suppose \( c(0) \) is an initial approximation to the solution of \( F(c(\lambda)) = 0 \).

Define \( G : [0,1] \times R_n \to R_n \) by
\[
G(\lambda, c) = \lambda F(c) + (1 - \lambda)[F(c) - F(c(0))]
\]
\[
= F(c) + (\lambda - 1)F(c(0))
\]

We will determine, for various values of \( \lambda \), a solution to \( G(\lambda, c) = 0 \). When \( \lambda = 0 \), this equation assumes the form \( 0 = G(0, c) = F(c) - F(c(0)) \), and \( c(0) \) is a solution.

When \( \lambda = 1 \), the equation assumes the form
\[
0 = G(1, c) = F(c),
\]
and \( c(1) = c^* \) is a solution. The function \( G \), with the parameter \( \lambda \), provides us with a family of functions that can lead from the known value \( c(0) \) to the solution \( c(1) = c^* \). The function \( G \) is called a homotopy between the function \( G(0, c) = F(c) - F(c(0)) \) and the function \( G(1, c) = F(c) \).

#### B. Continuation

According to [1], the continuation problem is used to determine away to proceed from the known solution \( c(0) \) of \( G(0, c) = 0 \) to the unknown solution \( c(1) = c^* \) of \( G(1, c) = 0 \), that is, to the solution to \( F(c) = 0 \). We first assume that \( c(\lambda) \) is the unique solution to the equation
\[
G(\lambda, c) = 0
\] (21)
for each \( \lambda \in [0,1] \). The set \( \{c(\lambda) \mid 0 \leq \lambda \leq 1\} \) can be viewed as a curve in \( R_n \) from \( c(0) \) to \( c(1) = c^* \) parameterized by \( \lambda \). A continuation method finds a sequence of steps along this curve corresponding to \( c(\lambda) \)
\[
k = 0, 1, \ldots, m, \quad \text{where} \quad \lambda_0 = 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_m = 1.
\]
If the functions \( \lambda \to c(\lambda) \) and \( G \) are differentiable, then differentiating (21) with respect to \( \lambda \) gives
\[
0 = \frac{\partial G(\lambda,c(\lambda))}{\partial \lambda} + \frac{\partial G(\lambda,c(\lambda))}{\partial c} c'(\lambda)
\]
Solving for \( c'(\lambda) \) gives
\[
c'(\lambda) = -\left[ \frac{\partial G(\lambda,c(\lambda))}{\partial c} \right]^{-1} \frac{\partial G(\lambda,c(\lambda))}{\partial \lambda}
\] (22)

Where
\[
\frac{\partial G(\lambda,c(\lambda))}{\partial c} = f(\lambda(c))
\]
is the Jacobian matrix, and
\[
\frac{\partial g(\lambda, c(\lambda))}{\partial \lambda} = F(c(0)).
\]

Therefore, the system of equation becomes:
\[
c'(\lambda) = -[f(\lambda(c))]^{-1}F(c(0)) \text{ for } 0 \leq \lambda \leq 1 \tag{23}
\]

with initial condition \(c(0)\).

To solve (23) using the Runge-Kutta method of order four, we first choose an integer \(N > 0\) and let \(h = (1 - 0)/N\). Partition the interval \([0, 1]\) into \(N\) subintervals with the mesh points 
\[\lambda_i = jh, \text{ for each } j = 0, 1, \ldots, N.\]

We use the notation \(w_{ij}\), for each \(j = 0, 1, \ldots, N\) and \(i = 1, \ldots, n\), to denote an approximation to \(c_i(\lambda_j)\).

For the initial conditions, set \(w_{1,0} = c_1(0), w_{2,0} = c_2(0), \ldots, w_{n,0} = c_n(0)\). Suppose \(w_{1,i}, w_{2,i}, \ldots, w_{n,i}\) have been computed. We obtain \(w_{1,i+1}, w_{2,i+1}, \ldots, w_{n,i+1}\) using the equations:
\[
k_{1,i} = h\varphi_i(\lambda_i, w_{1,i}, w_{2,i}, \ldots, w_{n,i}),
\]
for each \(i = 1, 2, \ldots, n\);
\[
k_{2,i} = h\varphi_i(\lambda_i + 0.5h, w_{1,i} + 0.5k_{1,i}, \ldots, w_{n,i} + 0.5k_{1,n}),
\]
for each \(i = 1, 2, \ldots, n\);
\[
k_{3,i} = h\varphi_i(\lambda_i + 0.5h, w_{1,i} + 0.5k_{2,i}, \ldots, w_{n,i} + 0.5k_{2,n}),
\]
for each \(i = 1, 2, \ldots, n\);
\[
k_{4,i} = h\varphi_i(\lambda_i + h, w_{1,i} + k_{3,i}, w_{2,i} + k_{3,2}, \ldots, w_{n,i} + k_{3,n}),
\]
for each \(i = 1, 2, \ldots, n\);

And finally,
\[
w_{1,i+1} = w_{1,i} + \frac{1}{6}(k_{1,i} + 2k_{2,i} + 2k_{3,i} + k_{4,i}), \text{ for each } i = 1, 2, \ldots, n.
\]

Let
\[
k_1 = \begin{bmatrix} k_{1,1} \\ k_{2,1} \\ \vdots \\ k_{1,n} \end{bmatrix}, \quad k_2 = \begin{bmatrix} k_{2,1} \\ k_{2,2} \\ \vdots \\ k_{2,n} \end{bmatrix}, \quad k_3 = \begin{bmatrix} k_{3,1} \\ k_{3,2} \\ \vdots \\ k_{3,n} \end{bmatrix}, \quad k_4 = \begin{bmatrix} k_{4,1} \\ k_{4,2} \\ \vdots \\ k_{4,n} \end{bmatrix}
\]
and
\[
w_j = \begin{bmatrix} w_{1,j} \\ w_{2,j} \\ \vdots \\ w_{n,j} \end{bmatrix}.
\]

\(c(0) = c(\lambda_0) = w_0\), and for each \(j = 0, 1, \ldots, N\), we have:
\[
k_1 = h[-f(w_j)]^{-1}F(c(0)),
\]
\[
k_2 = h[-f(w_j + 0.5k_1)]^{-1}F(c(0)),
\]
\[
k_3 = h[-f(w_j + 0.5k_2)]^{-1}F(c(0)),
\]
\[
k_4 = h[-f(w_j + k_3)]^{-1}F(c(0)),
\]
and
\[
c(\lambda_{i+1}) = c(\lambda_i) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4).
\]

Generally, \(c(\lambda_n) = c(1)\) is our approximation to \(c^*\).

IV. NUMERICAL EXAMPLES

A. Example 1:

Consider the second order nonlinear boundary value problem
\[
y'' + y^2 = x^2 \tag{24}
\]
With boundary conditions
\[
y(1) = 1, y(2) = 2 \tag{25}
\]
The exact solution is \(y(x) = x\).

Substituting (25) in (9) we get
\[
c_0 - c_1 + c_2 = 1\]
\[
c_0 + c_1 + c_2 = 2\]
For \(m = 2\) from (19) we get
\[
c_0^2 + \frac{1}{3}c_1^2 + 12c_2 - \frac{10217}{15}c_2^2 = \frac{7}{3}\]
After solving \(3 \times 3\) nonlinear system of algebraic equations (26) and (27) by homotopy continuation method we get
\[
c_0 = 1.5873, \ c_1 = 0.5 \text{ and } c_2 = -0.0873.
\]
Thus the approximate solution is
\[
\tilde{y}(x) = -0.5238x^2 + 2.5714x - 1.0476.
\]

| \(i\) | \(x\) | \(y\) | \(\tilde{y}\) | \(|y - \tilde{y}|\) |
|---|---|---|---|---|
| 0 | 1 | 1.0000 | 1.0000 | 0.0000 |
| 1 | 1.1 | 1.1000 | 1.1471 | 0.0471 |
| 2 | 1.2 | 1.2000 | 1.2838 | 0.0838 |
| 3 | 1.3 | 1.3000 | 1.4100 | 0.1100 |
| 4 | 1.4 | 1.4000 | 1.5257 | 0.1257 |
| 5 | 1.5 | 1.5000 | 1.6309 | 0.1309 |
| 6 | 1.6 | 1.6000 | 1.7257 | 0.1257 |
| 7 | 1.7 | 1.7000 | 1.8100 | 0.1100 |
| 8 | 1.8 | 1.8000 | 1.8838 | 0.0838 |
| 9 | 1.9 | 1.9000 | 1.9471 | 0.00471 |
| 10 | 2 | 2.0000 | 2.0000 | 0.0000 |

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B. Example 2:
Consider the second order nonlinear boundary value problem
\[ \frac{d^2y}{dx^2} + x \left( \frac{dy}{dx} \right)^2 = \frac{-1}{x^2} \]  
(28)

With boundary conditions
\[ y(1) = 0 , y(3) = \frac{2}{3} \]  
(29)

The exact solution is \( y(x) = 1 - \frac{1}{x} \)

Substituting (29) in (9) we get
\begin{align*}
&c_0 - c_1 + c_2 = 0 \\
&c_0 + c_1 + c_2 = \frac{2}{3}
\end{align*}  
(30)

For \( m = 2 \) from (19) we get
\[ 8c_1 + 28c_2 = -\frac{8}{9} \]  
(31)

After solving \( 3 \times 3 \) linear systems of algebraic equations (30) and (31) we get
\[ c_0 = 0.4603 , c_1 = 0.3333 \quad \text{and} \quad c_2 = -0.1270 \]

Thus the approximate solution is
\[ \tilde{y}(x) = -0.1905x^2 + 1.0953x - 0.9048 \]

C. Example 3:
Consider the nonlinear boundary value problem
\[ \frac{d^2y}{dx^2} + y \frac{dy}{dx} + y = \frac{1}{2} \sin(2x) \]  
(32)

With boundary conditions
\[ y(0) = 0 , y(\pi) = 0 \]  
(33)

The exact solution is \( y(x) = \sin x \)

Substituting (33) in (9) we get
\[ c_0 - c_1 + c_2 = 0 \\
&c_0 + c_1 + c_2 = 0 \]  
(34)

For \( m = 2 \) from (19) we get
\[ c_0\pi + 2c_0c_1\pi + 4c_1c_2\pi + 12c_2\pi = 0 \]  
(35)

After solving \( 3 \times 3 \) nonlinear systems of algebraic equations (34) and (35) with homotopy continuation method we obtain,
\[ c_0 = 0.7235 , c_1 = 0 \quad \text{and} \quad c_2 = -0.7235 \]

Thus the approximate solution is
\[ \tilde{y}(x) = 0.7235 - 0.7235(\frac{6x^2}{\pi^2} - \frac{6x}{\pi} + 1) \]

Table: Numerical Results of Example 3

| \( i \) | \( x \) | \( y \) | \( \tilde{y} \) | \( |y - \tilde{y}| \) |
|---|---|---|---|---|
| 0 | 0 | 0.0000 | 0.0000 | 0.0000 |
| 1 | 0.3 | 0.2955 | 0.3749 | 0.0794 |
| 2 | 0.6 | 0.5646 | 0.6707 | 0.1061 |
| 3 | 0.9 | 0.7833 | 0.8973 | 0.1040 |
| 4 | 1.2 | 0.9320 | 1.0247 | 0.0927 |
| 5 | 1.5 | 0.9975 | 1.0830 | 0.0855 |
| 6 | 1.8 | 0.9738 | 1.0621 | 0.0883 |
| 7 | 2.1 | 0.8632 | 0.9620 | 0.0988 |
| 8 | 2.4 | 0.6755 | 0.7828 | 0.1073 |
| 9 | 2.7 | 0.4274 | 0.5244 | 0.0970 |
| 10 | 3.0 | 0.1411 | 0.1868 | 0.0457 |

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V. CONCLUSION AND RECOMMENDATIONS

In this paper, the generalized shifted Legendre polynomial approximation on an arbitrary interval was designed to find an approximate solution of a given second order nonlinear two-point boundary value problems of ordinary differential equations. The advantages of this method is that it needs less computational time and effort and it can be applied to find the solution of second order nonlinear two-point boundary value problems defined on any arbitrary interval. Here problems whose exact solutions are different kinds of functions such as polynomials, rational, and trigonometric are considered. The results are illustrated by tables and graphs assisted by MATLAB. In the feature, the method may be extended to find the solutions of second order linear and nonlinear two-point boundary value problems of partial differential equations defined on any arbitrary intervals.

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