Computation of the Negative Damping Associated to the Hunting Motion of the Railway Wheelset, without Using Geometrical or Tribological Restrictions into the Model

Barenten Suciu

Abstract—Recently, analytical expressions for the damped natural frequency and damping ratio were proposed for the so-called dynamical hunting, either by assuming that the wheel conicity can be neglected, or by imposing restrictions on the ratio between the lateral and longitudinal creep coefficients, and also, on the ratio of the track span to the yawing diameter. However, instead of a pair of complex conjugate roots, and two real roots, of opposite sign, two pairs of complex conjugate roots were obtained for the characteristic equation. Purpose of this work is to achieve accurate expressions for the damping associated to the hunting motion, without imposing geometrical or tribological limitations into the vibration model, and to evaluate the error on the damping ratio, introduced by the simplified models. Also, nature of the roots of the characteristic equation is discussed, relative to the critical speed of the railway vehicle.

Index Terms—Railway Vehicle, Geometrical and Dynamical Hunting, Wheelset, Damping.

I. INTRODUCTION

Hunting vibration of the railway wheelset is described by a set of two differential equations, coupling the effects of the lateral and angular perturbations [1]–[6]. Damping effect [7]–[9] related to the hunting motion can be identified by analyzing the roots of the biquadratic characteristic equation [4]–[6]. Since the solutions of a quartic polynomial equation cannot be usually expressed by simple formulae [10]–[12], approximate methods were proposed to solve the hunting characteristic equation [3]–[6]. In one possible approach, by neglecting the inertial effects produced by the mass of the wheelset [3], the characteristic equation can be reduced to a quadratic equation. However, such model is unable to predict the dissipation effect occurring at the contact between wheels and rails [13]–[17]. Thus, in the absence of damping, it appears that once started, the vibration of the wheelset cannot be naturally halted, which disagrees with the hunting behavior observed on the actual carriages [4]–[5]. Alternatively, by neglecting the conicity of the wheels, the characteristic equation can also be reduced to quadratic type [6]. Further, according to Routh-Hurwitz stability conditions [18]–[20], the hunting mode appears as stabilized, under a critically-damped or over-damped contact type of the wheels with the rails [6]. Since such result was also in disagreement with the hunting vibration of many railway coaches, the quartic characteristic equation was next solved by imposing various restrictions on the ratio between the longitudinal and lateral creep coefficients, and also, on the ratio of the track span to the yawing diameter [4]–[5]. For example, although somewhat inaccurate, the assumptions that lateral creep coefficient almost equals the longitudinal creep coefficient [16]–[17], [21]–[23], and that the track span almost equals the yawing diameter [24]–[26], are widely accepted. Under such restrictions [4]–[5], some terms of the characteristic equation can be neglected, this leading to simpler analytical expressions for roots and damping ratio. Unfortunately, due to the alteration of the polynomial coefficients, instead of finding a pair of complex conjugate roots, and two real roots, of opposite signs, two pairs of complex conjugate roots were obtained. Naturally, this casts doubts upon the accuracy of the predicted damping at the contact of the wheels with rails.

Therefore, our aim is to derive reliable expressions for the damping associated to the hunting motion, without imposing geometrical and/or tribological limitations into the vibration model, and to evaluate the error introduced by the simplified models. Besides, we try to find the influence of the vehicle speed on the roots of the characteristic equation, and on the hunting damping ratio, this being of great interest especially for the design process of high-speed trains.

II. CHARACTERISTIC EQUATION ASSOCIATED TO THE HUNTING MOTION OF THE RAILWAY WHEELSET

By applying the Laplace transformation to the set of two differential equations of movement associated to the hunting vibration of the wheelset, the corresponding characteristic equation is obtained as a biquadratic polynomial equation, with a missing term in \( s^2 \) [5]–[6]:

\[
\begin{align*}
A_6 s^4 + A_5 s^3 + A_4 s^2 + A_3 s + A_2 &= 0 \\
A_4 &= 1 ; \quad A_1 = 0 \\
A_3 &= \frac{f \omega^2}{m R^2} + \frac{R^2}{V} = \omega_0 \left( \hat{b}^2 + \hat{f} \right) \\
A_2 &= \frac{f \omega^2}{m R^2} \frac{\hat{b} \hat{f}}{V} = \omega_0 \hat{b}^2 \hat{f} \\
A_1 &= \frac{f \omega^2}{m R^2} \frac{\hat{b}^3}{r} = \omega_0^2 \omega_1 \hat{b}^3 \hat{f}
\end{align*}
\]
where $s$ is the variable of the Laplace operator. Polynomial coefficients of (1) are depending on the natural circular frequency $\omega_s$ of the geometrical hunting motion, on the creep circular frequency $\omega_c$, on the creep ratio $\tilde{f}$, and on the dimensionless contact width $\tilde{b}$. These parameters can be calculated as [5]–[6]:

$$\omega_s = \sqrt{\frac{\lambda}{rb}}; \quad \omega_c = \frac{f_2}{mV}; \quad \tilde{f} = \frac{f_2}{f_1}; \quad \tilde{b} = \frac{b}{R_c} \tag{2}$$

where $m$ is the mass of the wheelset, $r$ is the wheel radius, $\lambda$ is the wheel conicity, $b$ is the track semi-span, $R_c$ is the yawing radius of gyration, $f_2$ is the lateral creep coefficient, $f_1$ is the longitudinal creep coefficient, and $V$ is the speed of the railway vehicle. Reference values of the geometrical and physical properties of the wheelset, wheels, and rails are given in Table I. Based on both theoretical [16]–[17] and experimental [21] studies on longitudinal and lateral creep coefficients, a variation range of $\tilde{f} = 0.7 - 1$ [5]–[6] can be accepted for the creep ratio (Table I). However, in order to simplify the analysis of the hunting motion, it is widely supposed that the lateral creep coefficient almost equals the longitudinal creep coefficient, and therefore, $\tilde{f} \approx 1$ [1]–[4].

Regarding the dimensionless contact width, it is also close to one ($\tilde{b} \approx 1$), for quite a large variety of carriages (Table I). Concerning the wheel conicity, higher values of 0.05–0.2 are typical for usual trains, and lower values of 0–0.025, implying nearly cylindrical wheels, are preferred for the high-speed trains.

### Table I: Geometrical and Physical Properties of the Wheelset, Wheels, and Rails

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass of the wheelset, $m$ [kg]</td>
<td>1.250 [25]</td>
</tr>
<tr>
<td>Wheel radius, $r$ [m]</td>
<td>0.43 [4]</td>
</tr>
<tr>
<td>Wheel conicity, $\lambda$ [-]</td>
<td>0.001; 0.025; 0.05; 0.1; 0.2 [1]–[6]</td>
</tr>
<tr>
<td>Track semi-span, $b$ [m]</td>
<td>0.745 [4]</td>
</tr>
<tr>
<td>Dimensionless contact width, $\tilde{b}$</td>
<td>$b/R_c$ [-]</td>
</tr>
<tr>
<td>Longitudinal creep coefficient, $f_1$ [N]</td>
<td>$1.5 \times 10^7$ [1]–[6]</td>
</tr>
<tr>
<td>Creep ratio, $\tilde{f}$</td>
<td>$f_2/f_1$ [-]</td>
</tr>
</tbody>
</table>

### III. APPROXIMATE COMPUTATION OF THE DAMPING RELATED TO THE HUNTING MOTION OF THE WHEELSET

Previously, by using a change of variable ($S = s + A_s/4$), (1) was firstly altered into a quartic polynomial equation $S^4 + B_3 S^3 + B_2 S + B_1 = 0$, with a missing term in $S^3$ [4]–[5]. Then, it was rewritten as a quadratic polynomial equation $\sigma^2 + B_3 \sigma + B_1 = 0 \ (\sigma = S^2)$, based on the observation that the coefficient $B_3 = 0.125 \omega_c (\tilde{b}^2 + \tilde{f}) (\tilde{b}^2 - \tilde{f})^2$ is almost nil, under the either stronger condition $\tilde{b} \geq \tilde{f} \geq 1$ [4], or weaker condition $(\tilde{b}^2 - \tilde{f})^2 \geq 0$ [5]. After the quadratic polynomial equation in $\sigma$ was solved, by reversing the changes of variables from $\sigma$ to $S$, and then, from $S$ to $s$, all the four solutions of (1) were attained, in the form of two pairs of complex conjugate roots. Then damping, naturally occurring during the hunting motion of the wheelset at the contact of the wheels with the rails, was identified based on similarities between the roots of (1) and the well-known solutions of the characteristic equation for a classical vibration system [5].

Thus, from the two pairs of complex conjugate roots, the following analytical expressions for the corresponding damping ratios $\zeta_1$ and $\zeta_2$ can be obtained:

$$\begin{align*}
\zeta_1 &= \frac{\Psi + \beta + 4 + 2 \sqrt{\Psi (\beta + 4)}}{2 \Psi - \beta + 2 \sqrt{\Psi (\beta + 4)}} > 0 \\
\zeta_2 &= \frac{\Psi + \beta + 4 - 2 \sqrt{\Psi (\beta + 4)}}{2 \Psi - \beta - 2 \sqrt{\Psi (\beta + 4)}} < 0 \tag{3}
\end{align*}$$

where the dimensionless parameters $\Psi$ and $\beta$ can be defined as:

$$\Psi = (\beta + 2)(\sqrt{1 + \Sigma} + 1) \geq 2; \quad \beta = \frac{(\tilde{b}^2 - \tilde{f})^2}{b^2 f^2} \geq 0 \tag{4}$$

in which the positive dimensionless parameter $\Sigma$ is depending on geometrical parameter $\tilde{b}$, on the tribological parameter $\tilde{f}$, and on the dimensionless hunting frequency $\Omega$, as follows:

$$\Sigma = \frac{64 \Omega^2}{(\beta + 2)^2} \geq 0; \quad \Omega^2 = \frac{\omega_c^2}{\omega_0^2 b^2 f^2} \geq 0 \tag{5}$$

Since under the either stronger condition $\tilde{b} \geq \tilde{f} \geq 1$ [4], or weaker condition $(\tilde{b}^2 - \tilde{f})^2 \geq 0$ [5], the parameter $\beta$ can be neglected ($\beta \approx 0$), after some simplifications carried on (4) and (5):

$$\Psi(\beta = 0) = 2(\sqrt{1 + \Sigma} - 1) \Leftrightarrow \Sigma = \Sigma(\beta = 0) = 16 \Omega^2 \tag{6}$$

quite simple analytical expressions for the damping ratios $\zeta_1$ and $\zeta_2$ can be obtained:

$$\begin{align*}
\zeta_1 &\approx \frac{1 + 1}{\sqrt{2}} \frac{2}{\sqrt{1 + 16 \Omega^2} + 1} > 0 \\
\zeta_2 &\approx -\frac{1 + 1}{\sqrt{2}} \frac{2}{\sqrt{1 + 16 \Omega^2} + 1} < 0 \tag{7}
\end{align*}$$

Note that for the positive damping ratio $\zeta_1$ of (3) or (7), the amplitude of vibration is decreasing as the time elapses. Therefore, such stable vibration mode should be disregarded. On the other hand, for the negative damping ratio $\zeta_2$ of (3) or (7), the amplitude of vibration is increasing as the time...
elapses. Since this vibration mode is able to describe the inherently unstable hunting motion of the wheelset, it is hereafter considered in the analysis.

Unfortunately, the abovementioned approach is based on the omission of one polynomial coefficient ($B_5 \neq 0$). As a result, instead of a pair of complex conjugate roots, and two real roots, of opposite signs, two pairs of complex conjugate roots were obtained. Since this result might have effect on the accuracy of predicted damping, the error introduced by such approximate computations is carefully evaluated, as shown in the next section.

IV. EXACT COMPUTATION OF THE DAMPING RELATED TO THE HUNTING MOTION OF THE WHEELSET

In this section, the characteristic equation (1) is accurately solved based on the method proposed by Ferrari [10]–[12], without imposing geometrical and/or tribological conditions into the vibration model.

A. Critical Speed of the Railway Vehicle

In order to determine the nature of the roots of the quartic polynomial equation, it is necessary to examine the sign of the parameters $\bar{A}$, $D$, and $P$ [10]–[12].

Firstly, the discriminant $\bar{A}$ can be written as [10]–[12]:

$$\bar{A} = 256A_0^3 - 128A_2^2A_0 + 144A_2^2A_4A_0 + $$
$$+ 16A_2^2A_0 - 27A_4^2A_2^2 - 4A_2^2A_0 =$$

$$= \omega_0^1B^2 \frac{f}{\Omega_i^2}[256\Omega_i^4 - (27\beta^2 + 72\beta - 16)\Omega_i^2 - 4\beta]$$

For a negative discriminant $\bar{A} < 0$ [10], attainable under the following condition:

$$0 \leq \Omega_i^2 < \frac{27\beta^2 + 72\beta - 16 + \sqrt{(\beta + 4)(9\beta + 4)^3}}{512}$$

(9)

two distinct real roots and a pair of two complex conjugate roots are to be obtained. As illustrated in the next section, such hunting pattern occurs at relatively lower speeds of the railway vehicle, and it is well-known in the literature [1]–[3]. Alternatively, for a positive discriminant $\bar{A} > 0$ [10]–[12], achievable under the following circumstances:

$$\Omega_i^2 > \frac{27\beta^2 + 72\beta - 16 + \sqrt{(\beta + 4)(9\beta + 4)^3}}{512}$$

(10)
either four real roots, or two pairs of complex conjugate roots, are to be obtained. In order to complete the analysis on the roots type, it is next necessary to examine the sign of the parameter $D$ [10]–[12]:

$$D = 64A_0^3 - 16A_2^2 + 16A_2^2A_4 - 3A_4^2 =$$

$$= \omega_0^1B^2 \frac{f}{\Omega_i^2}(64\Omega_i^4 - (3\beta^2 + 8\beta))$$

(11)

For a positive value $D > 0$ [10]–[11], acquirable under the following condition:

$$0 \leq \Omega_i^2 < \frac{3\beta^2 + 8\beta}{64}$$

(12)
a pair of two complex conjugate roots are to be obtained. As shown in the next section, such hunting pattern appears at relatively higher speeds of the railway vehicle, and it is similar to the hunting mode illustrated by the approximate approach of solving the characteristic equation.

On the other hand, for a negative value $D < 0$ [10]–[11], found in the following circumstances:

$$0 \leq \Omega_i^2 < \frac{3\beta^2 + 8\beta}{64}$$

(13)
in order to ascertain the type of the roots, it is necessary to examine the sign of the parameter $P$ [10]–[12]:

$$P = 8A_2 - 3A_4^2 = -\omega_0^1B^2 \frac{f}{\Omega_i^2}(3\beta + 4) < 0$$

(14)

From (14), parameter $P$ appears to be negative, regardless the actual value of $\beta \geq 0$, and in such circumstances, i.e., $D < 0$ and $P < 0$, all the four roots are real.

In order to easily settle the problem concerning the nature of the roots of the characteristic equation, it can be defined from the condition $\bar{A} = 0$, the critical dimensionless hunting circular frequency $\Omega_{tr,\bar{A}=0}$, as:

$$\Omega_{tr,\bar{A}=0} = \sqrt[4]{27\beta^2 + 72\beta - 16 + \sqrt{(\beta + 4)(9\beta + 4)^3}}$$

(15)

and its related critical speed of the railway vehicle, as:

$$V_{tr,\bar{A}=0} = \sqrt{r b f \bar{g} \frac{b^2 \bar{f}}{m^2 \lambda} \Omega_{tr,\bar{A}=0}^2}$$

(16)

Supplementary, it is useful to define from the condition $D = 0$, the critical dimensionless hunting circular frequency $\Omega_{tr,D=0}$, as:

$$\Omega_{tr,D=0} = \sqrt[4]{3\beta^2 + 8\beta}$$

(17)

and its associated critical speed of the railway vehicle, as:

$$V_{tr,D=0} = \sqrt{r b f \bar{g} \frac{b^2 \bar{f}}{m^2 \lambda} \Omega_{tr,D=0}^2}$$

(18)

Since the following inequality is satisfied regardless the actual value of the parameter $\beta$:

$$\frac{3\beta^2 + 8\beta}{64} \leq \frac{27\beta^2 + 72\beta - 16 + \sqrt{(\beta + 4)(9\beta + 4)^3}}{512}$$

(19)
one concludes that the critical hunting frequency $\Omega_{tr,D=0}$, and its associated speed $V_{tr,D=0}$, is always smaller or equal
to the critical hunting frequency $\Omega_{cr,3\omega_0}$, and its related speed $V_{cr,3\omega_0}$, i.e.:

$$\Omega_{cr,D=0} \leq \Omega_{cr,3\omega_0} ; \quad V_{cr,D=0} \leq V_{cr,3\omega_0} \quad (20)$$

In conclusion, for a velocity of the railway vehicle lower than the critical speed $V_{cr,3\omega_0}$, the characteristic equation displays two distinct real roots of opposite signs, and a pair of two complex conjugate roots, this being in agreement with the previously studied hunting pattern [1]–[3]. On the other hand, for speeds of the train higher than the critical speed $V_{cr,3\omega_0}$, the two real roots are changing into a pair of complex conjugate roots.

### B. Accurate Evaluation of the Damping Ratio

In order to accurately determine the damping ratio related to the hunting motion of the wheelset, firstly, based on the method proposed by Ferrari [10]–[12], the four roots of the characteristic equation (1) are written as:

$$s_{1,2} = -\frac{\omega_0 (b^2 + \tilde{f})}{4} - L \pm \frac{1}{2} \sqrt{4L^2 - 2p + \frac{q}{L}} \quad (21)$$

where the parameters of opposite signs $p = P/8$ (see (14)) and $q$ can be calculated as:

$$\begin{align*}
p &= -\omega_0^2 b^2 \tilde{f} (3\beta + 4) < 0 \\
q &= \omega_0^2 b^2 \tilde{f}^{3/2} \beta \sqrt{\beta + 4} \geq 0
\end{align*} \quad (22)$$

However, computation process of the parameter $L$ of (21) is more complicated, since it depends on the sign of the discriminant $\Delta$ (see (8)). Thus, for $\Delta \leq 0$ the parameter $L$ can be written as:

$$L = \frac{1}{2} \left( \frac{\omega_0^2 b^2 \tilde{f} (3\beta + 4)}{12} + \frac{1}{3} (Q + \tilde{\Delta}) \right) \quad (23)$$

where $\tilde{\Delta}$ is given by:

$$\tilde{\Delta} = \omega_0^2 \tilde{b}^2 \tilde{f}^2 (1 + 12\Omega^2) > 0 \quad (24)$$

and $Q$ can be calculated as:

$$Q = \sqrt{0.5(\Delta + \sqrt{-27\Delta})} \quad (25)$$

$$\Delta = \omega_0^2 \tilde{b}^2 \tilde{f}^2 [2 + 9\Omega^2 (3\beta + 4)] > 0$$

On the other hand, for $\tilde{\Delta} > 0$ the parameter $L$ is given by:

$$L = \frac{\omega_0 \tilde{b} \tilde{f}^{1/2}}{2} \left( \frac{3\beta + 4}{12} + \frac{2\sqrt{\Delta}}{3\omega_0^2 \tilde{b} \tilde{f}^2} + \cos \left( \frac{1}{3} \cos^{-1} \frac{0.5\Delta}{\Delta^{3/2}} \right) \right) \quad (26)$$

Next, in order to facilitate the damping ratio computation process, the following helpful functions are defined:

$$F_1(\beta) = 0.25\sqrt{\beta + 4} \quad (27)$$

$$F_2(\beta) = 0.125(3\beta + 4) \quad (28)$$

$$F_3(\beta) = 0.125 \sqrt{\beta + 4} \quad (29)$$

$$F_4(\Omega^2) = 1 + 12\Omega^2 \quad (30)$$

$$F_5(\beta, \Omega^2) = 2 + 9(3\beta + 4)\Omega^2 \quad (31)$$

$$F_6(\beta, \Omega^2) = \Omega^2 [256\Omega^4 - (27\beta^2 + 72\beta - 16)\Omega^2 - 4\beta] \quad (32)$$

$$F_7(\beta, \Omega^2) = \left[ \frac{1}{6} (F_1(\beta) + F_4^{1/2}(\Omega^2) \times \cos [\frac{1}{3} \cos^{-1} \frac{F_3(\beta, \Omega^2)}{2F_4^{1/2}(\Omega^2)}]) \right] \quad (33)$$

$$F_8(\beta, \Omega^2) = \left[ \frac{1}{6} F_2(\beta) + \frac{1}{12} F_3(\beta, \Omega^2) \right] \quad (34)$$

$$\tilde{F}_8(\beta, \Omega^2) = \sqrt{F_2(\beta, \Omega^2) + \sqrt{-27F_8(\beta, \Omega^2)}} \quad (35)$$

$$F_9(\beta, \Omega^2) = -4F_7^{1/2}(\beta, \Omega^2) + 2F_2(\beta) + \frac{F_3(\beta)}{F_4(\beta, \Omega^2)} \quad (36)$$

$$F_{10}(\beta, \Omega^2) = -4F_7^{1/2}(\beta, \Omega^2) + 2F_2(\beta) - \frac{F_3(\beta)}{F_4(\beta, \Omega^2)} \quad (37)$$

In such conditions, the various parameters composing the expressions of the roots of the characteristic equation (see (21)-(26) can be compactly rewritten as:

$$\omega_0 \tilde{b} \tilde{f}^{1/2} = \omega_0 \tilde{b} \sqrt{\tilde{f}^2} \times F_1(\beta) \quad (38)$$

$$p = -\omega_0^2 \tilde{b}^2 \tilde{f}^2 \cdot F_2(\beta) \quad (39)$$

$$q = \omega_0^2 \tilde{b}^2 \tilde{f}^{3/2} \times F_3(\beta) \quad (40)$$

$$\tilde{\Delta} = \omega_0^2 \tilde{b}^2 \tilde{f}^2 \times F_4(\Omega^2) \quad (41)$$

$$\Delta = \omega_0^2 \tilde{b}^2 \tilde{f}^2 \times F_5(\beta, \Omega^2) \quad (42)$$

$$\tilde{\Delta} = \omega_0^2 \tilde{b}^2 \tilde{f}^2 \times \Delta \quad (43)$$

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$$L = \omega \bar{b} \sqrt{f \times \left[ F_i(\beta, \Omega^2) ; \bar{b} > 0 \right.}$$
$$\left. F_i(\beta, \Omega^2) ; \bar{b} \leq 0 \right]$$

(44)

Consequently, solutions (21) of the characteristic equation can be rewritten as:

$$s_{1,2} = \omega \bar{b} \tilde{f}^{1/2} \left[ -F_i(\beta) - F_{3,8}(\beta, \Omega^2) \pm \frac{1}{2} \sqrt{F_i(\beta, \Omega^2)} \right]$$
$$s_{3,4} = \omega \bar{b} \tilde{f}^{1/2} \left[ -F_i(\beta) + F_{3,8}(\beta, \Omega^2) \pm \frac{1}{2} \sqrt{F_{10}(\beta, \Omega^2)} \right]$$

(45)

Next damping, spontaneously occurring during hunting motion of the wheelset, is identified based on the similarities between the roots (45), and the well-known solutions of the characteristic equation for a classical vibration system [5]. Firstly, the accurate damping ratio \( \zeta_{1,a} \), related to the roots \( s_{1,2} \) of (21), appears to be positive, and can be calculated as:

$$\zeta_{1,a} = \frac{2(F_i + F_{3,8})}{\sqrt{4(F_i + F_{3,8})^2 + |F_{10}|}} > 0$$

(46)

Then, the accurate damping ratio \( \zeta_{2,a} \), corresponding to the roots \( s_{3,4} \) of (21), can be estimated as:

$$\zeta_{2,a} = \frac{2(F_i - F_{3,8})}{\sqrt{4(F_i - F_{3,8})^2 + |F_{10}|}} < 0$$

(47)

Arguments \( \beta \) and \( \Omega^2 \) of the functions \( F_i \), \( F_{3,8} \), \( F_9 \), and \( F_{10} \) are omitted in (46)-(47), for compact description of the damping ratios.

Again, for the positive damping ratio \( \zeta_{1,a} \) of (46), the vibration amplitude reduces as the time passes, and such stable vibration mode is disregarded in our analysis.

Concerning (47), it can be demonstrated that \( F_i < F_{3,8} \), regardless the values of \( \beta \) and \( \Omega^2 \). Thus, the condition \( F_i > F_9 \) can be satisfied only under the restriction \( \beta < -4 \), which cannot be achieved from a practical point of view, since generally \( \beta \geq 0 \) (see (4)). On the other hand, \( F_i > F_9 \) leads to \((\tilde{F}_i - 1)^2 + 12\Omega^2 < 0\), an inequality which cannot be satisfied, because \((\tilde{F}_i - 1)^2 \geq 0 \) and \(12\Omega^2 \geq 0\).

Since \( (F_i - F_{3,8}) < 0 \), it seems that the damping ratio \( \zeta_{2,a} \) of (47) is generally negative, i.e., the hunting motion of the wheelset is inherently unstable.

V. RESULTS AND DISCUSSIONS

A. Variation Range of the Dimensionless Parameter \( \beta \)

Fig. 1 shows the variation of the dimensionless parameter \( \beta \) versus the creep ratio \( \tilde{f} \), associated to various values for the dimensionless contact width, i.e., \( \tilde{b} = 0.9, 0.95, 1, 1.05, \) and 1.1. Fig. 1 illustrates that the minimal value \( \beta_{\min} = 0 \) is reached for \( \tilde{b}^2 = \tilde{f} \), and this result can be confirmed by examining the partial derivatives of \( \beta \) with respect to variables \( \tilde{b} \) and \( \tilde{f} \):

$$\frac{\partial \beta}{\partial \tilde{b}} = \frac{2(\tilde{b}^4 - \tilde{f}^2)}{\tilde{b}^4 \tilde{f}^2} ; \frac{\partial \beta}{\partial \tilde{f}} = \frac{\tilde{f}^2 - \tilde{b}^4}{\tilde{b}^4 \tilde{f}^2}$$

(48)

On the other hand, the maximal value \( \beta_{\max} \approx 0.3 \) is attained for \( \tilde{b} = 1.1 \) and \( \tilde{f} = 0.7 \) (see Fig. 1).

![Fig. 1. Variation of the dimensionless parameter beta versus the creep ratio, for various values of the dimensionless contact width.](image1)

B. Variation of the Critical Hunting Circular Frequency

Fig. 2 illustrates the monotonical nonlinear augmentation of the critical dimensionless circular frequencies, \( \Omega_{cr,\bar{b} = 0} \) and \( \Omega_{cr,\bar{b} = 0} \), versus the dimensionless parameter \( \beta \).

Numerical results agree with the theoretically predicted inequality (20). Moreover, the variation pattern indicated by Fig. 2, is consistent with positive derivatives of the critical dimensionless circular frequencies, with respect to \( \beta \):

$$\frac{\partial \Omega_{cr,\bar{b} = 0}}{\partial \beta} > 0 ; \frac{\partial \Omega_{cr,\bar{b} = 0}}{\partial \beta} > 0$$

(49)

![Fig. 2. Variation of the critical dimensionless circular frequencies versus the dimensionless parameter beta.](image2)
Two distinct hunting modes can be discerned (see Fig. 2), since the characteristic equation has in the hunting mode corresponding to lower frequencies \( (\Omega < \Omega_{\text{cr},3,0}) \), two real and two complex conjugate roots, but in the hunting mode associated to higher frequencies \( (\Omega > \Omega_{\text{cr},3,0}) \), two pairs of complex conjugate roots. Nature of the roots seems to be decided solely in relation with the frequency \( \Omega_{\text{cr},3,0} \) and consequently, the other critical frequency \( \Omega_{\text{cr},3,0} \) can be disregarded in the following analysis.

C. Variation of the Critical Speed of the Railway Vehicle

Fig. 3 shows the variation of the critical speed of the railway vehicle \( V_{\text{cr},3,0} \) versus the creep ratio \( \bar{f} \), associated to various values for the dimensionless contact width, i.e., \( \bar{b} = 0.9, 0.95, 1, 1.05, \) and 1.1. These results were obtained by taking the following numerical values for the track semi-span, \( b = 0.745 \text{ m} \); wheel radius, \( r = 0.43 \text{ m} \); wheel conicity, \( \lambda = 0.025 \); mass of the wheelset, \( m = 1.250 \text{ kg} \); as well as the longitudinal creep coefficient \( f_1 = 1.5 \times 10^7 \text{ N} \) (Table I).

A valley-type variation pattern can be generally observed for the critical speed of the railway vehicle (see the results for \( \bar{b} = 0.9 \) and 0.95). Bottom of the valley is reached for \( \bar{b}^2 = \bar{f} \) (see the results for \( \bar{b} = 0.9 - 1 \)). However, for the larger dimensionless contact widths \( \bar{b} = 1 - 1.1 \), due to the upper limitation of the creep ratio to the value \( \bar{f} = 1 \), only the descending part of the valley can be observed.

Again, the characteristic equation has in the hunting mode corresponding to lower velocities \( (V < V_{\text{cr},3,0}) \), two real and two complex conjugate roots, but in the hunting mode associated to higher velocities \( (V > V_{\text{cr},3,0}) \), two pairs of complex conjugate roots.

D. Variation of the Approximate Damping Ratio

Fig. 4 displays the variation of the approximate damping ratio \( \zeta_2 \) of (3) versus the dimensionless hunting circular frequency \( \Omega \), which is proportional to the square speed \( V^2 \) of the railway vehicle (see (2) and (5)). One observes two distinct patterns of variation for the approximate damping ratio. Thus, for \( \beta = 0 \), a monotonical nonlinear decreasing pattern, and for \( \beta = 0.1 - 0.3 \), a mountain-like pattern of variation can be discerned. However, for large values of the hunting circular frequency \( \Omega \), the graphs are superposed for all values of the parameter \( \beta \). In order to explain these findings, one firstly observes that the monotonical nonlinear decreasing pattern is related to the hunting corresponding to \( \Omega > \Omega_{\text{cr},3,0} \). Since a nil critical frequency \( \Omega_{\text{cr},3,0} = 0 \) is corresponding to \( \beta = 0 \) (see (15)), for any value of the hunting circular frequency \( \Omega \), the characteristic equation has two pairs of complex conjugate roots. On the other hand, for \( \beta = 0.1 - 0.3 \), the ascending part of the mountain-like graph corresponds to \( \Omega < \Omega_{\text{cr},3,0} \), but the descending part is related to \( \Omega > \Omega_{\text{cr},3,0} \). We recall here that, in the simplified method (3), due to the alteration of the coefficients of the characteristic equation, instead of finding a pair of complex conjugate roots, and two real roots, two pairs of complex conjugate roots were obtained. Naturally, this casts doubts upon the damping ratio displayed by the ascending part of the mountain-like graphs, and this aspect will be clarified in the next paragraph.

For carriages running at very high speeds \( (V \rightarrow \infty) \), i.e., for \( \Omega \rightarrow \infty \), the hunting damping ratio approaches the limit \(-1/\sqrt{2} \), a result which agrees with the findings of [5].

E. Variation of the Accurate Damping Ratio

Fig. 5 presents the variation of the accurate damping ratio \( \zeta_2 \) of (47) versus the dimensionless hunting circular frequency \( \Omega \), for various values of the parameter \( \beta \). One cannot distinguish significant difference between the results obtained for various values of \( \beta \), taken in the range of \( 0 - 0.3 \). Based on Fig. 5 one concludes that the accurate damping ratio is actually unaffected by the specific hunting mode, corresponding either to \( \Omega < \Omega_{\text{cr},3,0} \) or \( \Omega > \Omega_{\text{cr},3,0} \). Besides, it seems that the ascending parts of the mountain-like graphs from Fig. 4, cannot be accepted as

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reliable approximations for the damping ratio. Surprisingly, for coaches running at very high speeds \((V \to \infty)\), i.e., for \(\Omega \to \infty\), similar to the variation pattern of the approximate damping ratio (see Fig. 4), the accurate damping ratio approaches the same limit \(-1/\sqrt{2}\) (Fig. 5).

![Graph](image)

**Fig. 5.** Variation of the accurate damping ratio versus the dimensionless hunting circular frequency, \(\Omega\) [-].

**F. Comparison between the Approximate and Accurate Damping Ratio**

Fig. 6 illustrates the variation of the approximate damping ratio \(\zeta_a\) of (3), as well as the variation of the accurate damping ratio \(\zeta_{2,a}\) of (47) versus the dimensionless hunting circular frequency \(\Omega\), for various values of the parameter \(\beta = 0, 0.1, 0.2, \) and \(0.3\). Based on Fig. 6 one concludes that neither the actual hunting mode, nor the parameter \(\beta\) have influence on the damping ratio associated to the hunting motion of the wheelset.

![Graph](image)

**Fig. 6.** Variation of the approximate and accurate damping ratios versus the dimensionless hunting circular frequency, for various values of the parameter beta.

Moreover, instead of employing a complex computation process (27)-(47), the negative damping ratio \(\zeta\), occurring spontaneously during the hunting movement of the wheelset, can be calculated with a relative error smaller than 5% by the following formula, which is quite easy to be used in the design process of the railway vehicles:

\[
\zeta = \frac{1 - \frac{1}{2} \left( \frac{2}{\sqrt{1 + 16\Omega^2} + 1} \right)}{1 - \frac{1}{2} \left( \frac{2}{\sqrt{1 + 16m^2\lambda R^2} + 1} \right)}
\]

(50)

**VI. CONCLUSION**

In this work a simple but reliable relationship to compute the negative damping ratio associated to the hunting motion of the wheelset was achieved, without imposing geometrical or tribological limitations into the vibration model. In order to diminish the effects of the wheelset unstable hunting mode, the absolute value of the associated damping ratio should be decreased, as much as possible. This can be technically achieved by:

1. Augmentation of the wheel radius \(r\) and/or the span \(b\) of the track. Larger effect is expected by changing the track span, due to its second power influence. However, normally, the designer cannot alter these standardized dimensions of the railway transportation system.

2. Augmentation of the longitudinal \(f_1\) and/or lateral \(f_2\) creep coefficients. This can be achieved either by increasing the total mass of the carriage, or by altering the material properties of the wheels and railways. Nevertheless, the designer cannot easily modify the specifications of the standardized steels employed by the railway transportation system.

3. Reduction of the mass \(m\) and/or yawing radius \(R\) of gyration of the wheelset. Again, larger effect is obtained by changing the mass, due to its second power influence.

4. Reduction of the wheel conicity \(\lambda\), which seems to be a viable option, since for cylindrical wheels (\(\lambda = 0\)) the damping ratio becomes nil, and consequently, the hunting motion stabilizes.

5. Reduction of the travelling speed of the railway vehicle. However, this cannot be considered as a viable option for the modern high-speed transportation systems.

6. Addition of positive damping into the system, by using yaw dampers. Therefore, by achieving a total positive damping ratio, effects of the inherently unstable hunting mode can be counteracted.

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