Introduction into the Extra Geometry of the Three–Dimensional Space I

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***Abstract*—Using the theory of exploded numbers by the axiom–systems of real numbers and Euclidean geometry, we introduce an extra geometry in the three–dimensional space. In this section we deal with extra-lines corresponding to Euclidean lines. Extra-lines can be curved in Euclidean sense. Moreover, extra geometry differs from the Euclidean geometry in term of the parallelism of the lines.**

***Index Terms*—Border Points, Exploded and Compressed Numbers, Extra – Line, Extra Parallelism, Super – Line**

I. The Explosion of the Three–Dimensional Space

The concept of exploded numbers had already been introduced in [1] and the explosion of *k*– dimensional space was discussed in the 4th part of [2]. In this article $k = 1\,,2\ and\ 3$ are used, only.

For the sake of a better understanding we repeat some facts. Denoting by $\mathbb{R}$ the set of real numbers, we repeat the axioms of real numbers: commutativity and associativity of both addition and multiplication, existence of unit elements for addition and multiplication (0 and 1, respectively) existence of additive inverse element $(-a)$ where $a \in \mathbb{R}$, and multiplicative inverse element of $a \in \mathbb{R}$ ($\frac{1}{a}$, where $a \neq 0$), and distributivity. Summarizing we have the field $(\mathbb{R}\,,+,\cdot\,,=)$. Moreover, we have the relation „$<$" and its monotonity for addition and multiplication (if $a < b$ then $a + c < b + c$ and assuming that $c > 0$ we have that $a \cdot c < b \cdot c$), so we say that $(\mathbb{R}\,,+,\cdot\,,\leq)$ is an ordered field. Finally, we mention that the completeness axiom: every nonempty proper subset $\mathbb{S}$ of the real numbers $\mathbb{R}$ has the least upper bound $x \in \mathbb{R}$. ($x \geq y$ for all $y \in \mathbb{S}$ such that if z is an upper bound for $\mathbb{S}$ then $x \leq z$). The construction of this axiom-system is due to Dedekind (1831-1916). He developed the idea first in 1859 though he did not publish it until 1872. The final result is seen e.g in [5]. Moreover, *we use the following postulates and requirements.* Let σ be a fixed positive number.

Postulate of extension: The set of real numbers $\mathbb{R}$ is a proper subset of the set of exploded real numbers $\breve{\mathbb{R}}^\sigma$. For any real number *x* there exists only one exploded number $\breve{x}^\sigma$ called exploded *x* or the exploded number of *x*. On the other hand *x* is called the compressed number of $\breve{x}^\sigma$ that is

$$\underline{(\breve{x}^\sigma)}_\sigma = x \;, \qquad x \in \mathbb{R},$$

where the compression is denoted by $\underline{\ \ }_\sigma$. Moreover, for any $u \in \breve{\mathbb{R}}^\sigma$ there exists only one real number, which is called the compressed number of *u* and denoted by $\underline{u}_\sigma$, such that

$$\breve{(\underline{u}_\sigma)}^\sigma = u\,, u \in \breve{\mathbb{R}}^\sigma$$

Postulate of unambiguity: For any pair of real numbers *x* and *y*, their exploded numbers are equal (in $\breve{\mathbb{R}}^\sigma$.) if and only if *x* is equal to *y* (in $\mathbb{R}$). Shortly,

$$\breve{x}^\sigma \breve{=} \breve{y}^\sigma \Leftrightarrow x = y$$

Postulate of ordering: For any pair of real numbers *x* and y, the exploded *x* is smaller (in $\breve{\mathbb{R}}^\sigma$) than exploded *y* if and only if *x* smaller than *y* (in $\mathbb{R}$). Shortly,

$$\breve{x}^\sigma \breve{<} \breve{y}^\sigma \Leftrightarrow x < y$$

Postulate of super–addition: For any pair of real numbers *x* and *y*, the super-sum of their exploded number is the exploded of their sum. To put it by symbols:

$$\breve{x}^\sigma \overline{\oplus}^\sigma \breve{y}^\sigma = \breve{(x+y)}^\sigma$$

Postulate of super–multiplication: For any pair of real numbers *x* and *y* , the super-product of their exploded numbers is the exploded number of their product. Expressed by symbols:

$$\breve{x}^\sigma \overline{\odot}^\sigma \breve{y}^\sigma = \breve{(x \cdot y)}^\sigma$$

Requirement of equality for exploded real numbers: If *x* and *y* are real numbers then *x* as an exploded number equals to *y* as an exploded number if and only if they are equal in the traditional sense. Shortly,

$$x \breve{=} y \Leftrightarrow x = y$$

Requirement of ordering for exploded real numbers: If *x* and *y* are real numbers then *x* as an exploded number is smaller than *y* as an exploded number if and only if *x* is smaller than *y* in the traditional sense. Shortly,

$$x \breve{<} y \Leftrightarrow x < y$$

After the requirements of equality and ordering, distinguishing between the equalities and orderings in $\breve{\mathbb{R}}^c$ and $\mathbb{R}$ is unnecessary, so instead of „$\breve{=}$" we can write „=" and instead of „$\breve{<}$" we can write „<".

Requirement for zero: The exploded number of 0 is itself. Expressed by symbols

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$$\breve{0}^\sigma = 0$$

After the latter requirement the following definitions are given:

The exploded number $\breve{x}^\sigma$ is called positive if $0 < \breve{x}^\sigma$.

(By the Postulate of ordering it is fulfilled if and only if $x > 0$.)

The exploded number $\breve{x}^c$ is called negative if $\breve{x}^\sigma < 0$.

(By the Postulate of ordering it is fulfilled if and only if $x < 0$.)

Requirement of monotonity of super–addition: If $\breve{x}^\sigma$ and $\breve{y}^\sigma$ are arbitrary exploded numbers and $\breve{x}^\sigma$ is smaller than $\breve{y}^\sigma$ then, for any exploded number $\breve{z}^\sigma$, the super-sum $\breve{x}^\sigma \overline{\oplus}^\sigma \breve{z}^\sigma$ is smaller than super-sum $\breve{y}^\sigma \overline{\oplus}^\sigma \breve{z}^\sigma$.

Requirement of monotonity of super–multiplication: If $\breve{x}^\sigma$ and $\breve{y}^\sigma$ are arbitrary exploded numbers and $\breve{x}^\sigma$ is smaller than $\breve{y}^\sigma$ then, for any positive exploded number $\breve{z}^\sigma$, the super-product $\breve{x}^\sigma \overline{\odot}^\sigma \breve{z}^\sigma$ is smaller than super-product $\breve{y}^\sigma \overline{\odot}^\sigma \breve{z}^\sigma$.

By isomorphism

$$x \leftrightarrow \breve{x}^\sigma \quad , x \in \mathbb{R}$$

We can find that the set of exploded real number $\breve{\mathbb{R}}^\sigma$ is an ordered field with respect to super-addition and super-multiplication. Although $\breve{\mathbb{R}}^\sigma$ is isomorphic with the set of real numbers $\mathbb{R}$, but super-operations are not extensions of traditional operations.

For the sake of comfort, we use the hyperbolic exploder function (see [3], (4.1) page 41) and say

$$\breve{x}^\sigma = \sigma \cdot \tanh^{-1}\frac{x}{\sigma} = \frac{\sigma}{2} \cdot ln\frac{1-\frac{x}{\sigma}}{1+\frac{x}{\sigma}}, \text{ where } -\sigma < x < \sigma \qquad (1)$$

If $-\infty < x \leq -\sigma$ or $\sigma \leq x < \infty$ then we say that $\breve{x}^\sigma$ is an *invisible exploded number*. By the Postulate of ordering we have that the $\widetilde{(-\sigma)}^\sigma$ is the greatest invisible exploded number which is less than each real number, called *negative discriminator*. The $\breve{\sigma}^\sigma$ is the smallest invisible exploded number which is greater than each real number, called *positive discriminator*.

By the Postulate of extension, the explosion formula (1) yields the compression formula

$$\underline{x}_\sigma = \sigma \cdot \tanh\frac{x}{\sigma} = \sigma \cdot \frac{e^{\frac{x}{\sigma}} - e^{-\frac{x}{\sigma}}}{e^{\frac{x}{\sigma}} + e^{-\frac{x}{\sigma}}}, \text{ where } -\infty < x < \infty \qquad (2)$$

The compression formula (2) shows that if $x \in \mathbb{R}$ then $-\sigma < \underline{x}_\sigma < \sigma$.

The exploded two–dimensional and three–dimensional spaces are

$$\widetilde{\mathbb{R}^2}^\sigma = \{(\breve{x}^\sigma, \breve{y}^\sigma) | (x, y) \in \mathbb{R}^2\} \qquad (3)$$

and

$$\widetilde{\mathbb{R}^3}^\sigma = \{(\breve{x}^\sigma, \breve{y}^\sigma, \breve{z}^\sigma) | (x, y, z) \in \mathbb{R}^3\} \qquad (4)$$

respectively. Of course $\mathbb{R}^2 \subset \widetilde{\mathbb{R}^2}^\sigma$ and $\mathbb{R}^3 \subset \widetilde{\mathbb{R}^3}^\sigma$. Moreover, the exploded spaces have invisible points, too. In general, if the set $\mathbb{S} \subseteq \mathbb{R}^3$, then

$$\breve{\mathbb{S}}^\sigma = \{(\breve{x}^\sigma, \breve{y}^\sigma, \breve{z}^\sigma) | (x, y, z) \in \mathbb{S}\} \qquad (5)$$

Of course if one of the coordinates $\breve{x}^\sigma, \breve{y}^\sigma or\ \breve{z}^\sigma$ is invisible exploded number, then the point $(\breve{x}^\sigma, \breve{y}^\sigma, \breve{z}^\sigma)$ is invisible in the three–dimensional space $\mathbb{R}^3$.

If the set $\mathbb{S} = \{(u, v, w) | u, v, w \in \breve{\mathbb{R}}^\sigma\} \subseteq \widetilde{\mathbb{R}^3}^\sigma$, then

$$\underline{\mathbb{S}}_\sigma = \{(\underline{u}_\sigma, \underline{v}_\sigma, \underline{w}_\sigma) | (u, v, w) \in \mathbb{S}\} \qquad (6)$$

Of course $\underline{\mathbb{S}}_\sigma \subseteq \mathbb{R}^3$. In the special case $\mathbb{S} = \widetilde{\mathbb{R}^3}^\sigma$, the compression (6) says that

$$\underline{\left(\widetilde{\mathbb{R}^3}^\sigma\right)}_\sigma = \mathbb{R}^3$$

On the other hand if $\mathbb{S} = \mathbb{R}^3$ then by (6)

$$\underline{\mathbb{R}^3}_\sigma = \left\{(x, y, z) \in \mathbb{R}^3 \middle| \begin{cases} -\sigma < x < \sigma \\ -\sigma < y < \sigma \\ -\sigma < z < \sigma \end{cases}\right\} \qquad (7)$$

is obtained. So, $\underline{\mathbb{R}^3}_\sigma$ is an open cube in $\mathbb{R}^3$. Moreover

$$\widetilde{\left(\underline{\mathbb{R}^3}_\sigma\right)}^\sigma = \mathbb{R}^3$$

is valid.

If the set $\mathbb{S} \subseteq \widetilde{\mathbb{R}^3}^\sigma$ then it may have invisible points. For the visible points of $\mathbb{S}$ we introduce the concept of *box–phenomenon*.

$$\mathbb{S}_{box} = \mathbb{S} \cap \mathbb{R}^3 \qquad (8)$$

Of course the box–phenomenon may be the empty set, too. If *x* a fixed real number then the visibility of $\breve{x}^\sigma$ depends on σ. If $|x| \geq \sigma$ then $\breve{x}^\sigma$ is invisible. If $|x| < \sigma$ then a familiar discussion (see e.g. [6]) of the function $\tau_x(\sigma) = \sigma \cdot \tanh^{-1}\frac{x}{\sigma}$ gives:

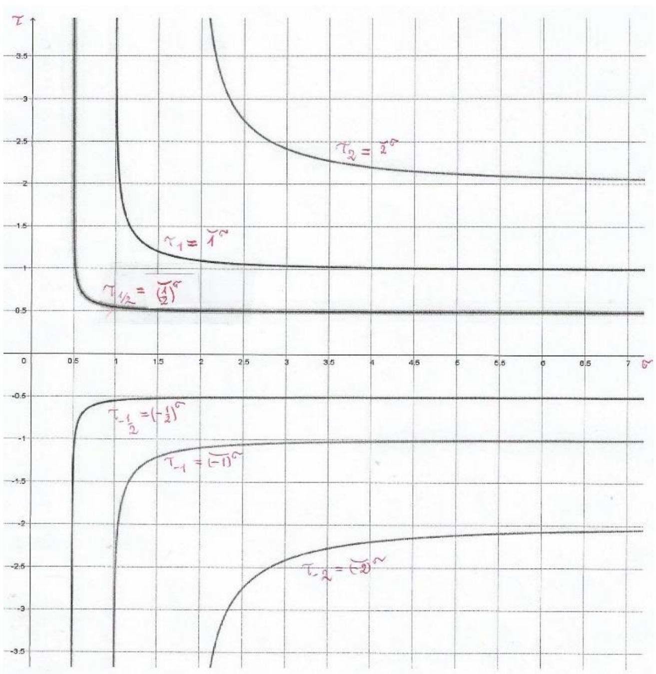

Fig. 1.

We can see the cases $x = \pm\frac{1}{2}, \pm 1$ and ± 2.

By the Requirement for zero we know that the super addition unit $\breve{0}^{\sigma}$ is visible for any σ. On the other hand the discriminators $\breve{\sigma}^{\sigma}$ and $\breve{-\sigma}^{\sigma}$ are invisible for any σ. The Fig. 1 illustrates this fact by the limits of functions τ_x .

$$\lim_{\substack{\sigma\to x(>0)\\ \sigma>x}} \tau_x(\sigma) = \infty \quad \text{and} \quad \lim_{\substack{\sigma\to -x(>0)\\ \sigma>-x}} \tau_x(\sigma) = -\infty,$$

in the cases $x = \pm\frac{1}{2}, \pm 1$ and ± 2, respectively. On the other hand, we can see that

$$\lim_{\sigma\to\infty} \breve{x}^{\sigma} = x \;\text{ such that if } x \neq 0 \text{ then } |\breve{x}^{\sigma}| > |x|$$

Especially interesting the case of the super - multiplication unit $\breve{1}^{\sigma}$. If $0 < \sigma \leq 1$ then $\breve{\sigma}^{\sigma} \leq \breve{1}^{\sigma}$. By the Postulate of ordering the exploded number $\breve{1}^{\sigma}$ is greater then or equal the positive discriminator and it is invisible. If $\sigma > 1$, then by (1) we have $\breve{1}^{\sigma} = \sigma \cdot \tanh^{-1}\frac{1}{\sigma} \in \mathbb{R}$, so it is a visible exploded number (See $\tau_1(\sigma)$ in Fig. 1).

Let now x be an arbitrary but fixed real number. The familiar discussion of the function

$$\rho_x(\sigma) = \sigma \cdot \tanh\frac{x}{\sigma} \quad , \quad 0 < \sigma < \infty,$$

We get its graph

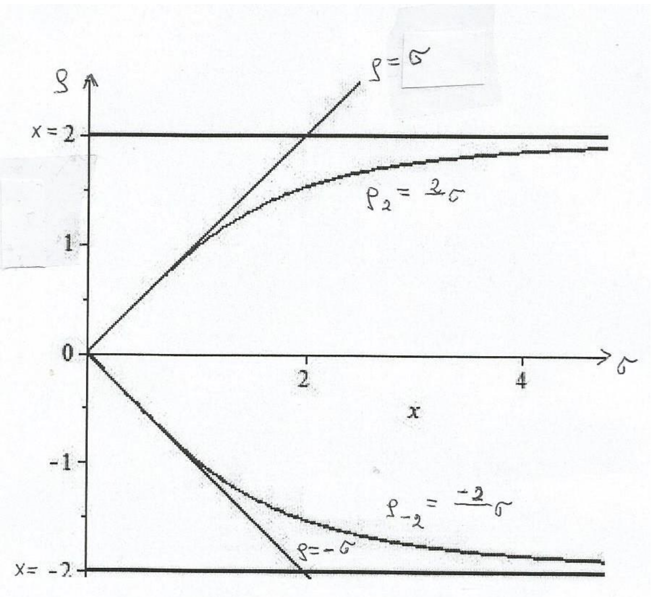

Fig. 2.

Fig. 2 shows the limit

$$\lim_{\substack{\sigma\to 0\\ \sigma>0}} \rho_x(\sigma) = 0$$

which says the the construction to define the set $\underline{\mathbb{R}^3}_0$ is unsuccesful, because it would be the empty set (See (7)). Moreover, Fig. 2 shows that $\lim_{\sigma\to\infty} \rho_x(\sigma) = x$. We can see that

$$\lim_{\sigma\to\infty} \underline{x}_{\sigma} = x \text{ such that if } x \neq 0 \text{ then } |\underline{x}_{\sigma}| < |x|.$$

II. Extra-Lines in the Three – Dimensional Space

For sake of simplicity, we use $\sigma = 1$. Moreover, the explosion and compression will be denoted without the mention σ. For example, instead $\breve{x}^1$ or $\underline{x}_1$ we write $\breve{x}$ and $\underline{x}$, respectively. So, the explosion and compression formulas under (1) and (2) become

$$\breve{x} = \tanh^{-1} x \quad , \; -1 < x < 1 \tag{9}$$

and

$$\underline{x} = \tanh x \quad , \quad -\infty < x > \infty \tag{10}$$

Similarly, instead of (7) we use

$$\underline{\mathbb{R}^3} = \left\{(x,y,z) \in \mathbb{R}^3 \middle| \begin{cases} -1 < x < 1 \\ -1 < y < 1 \\ -1 < z < 1 \end{cases}\right\} \tag{11}$$

Let $P_0 = (x_0, y_0, z_0) \in \underline{\mathbb{R}^3}$ and $E = (e_x, e_y, e_z) \in \mathbb{R}^3$ be a given point and vector, such that $\|E\| = 1$, respectively. We consider the straight line $\mathbb{L}_{P_0,E}$ of Euclidean geometry (see e.g. [7]). It is known that $\mathbb{L}_{P_0,E}$ is a good approximation for the way of the light in the case of short distances. $\mathbb{L}_{P_0,E}$ is given by the vector-equation

$$P_t = P_0 + t \cdot E \quad , -\infty < t < \infty \tag{12}$$

where $P_t = (x_t, y_t, z_t) \in \mathbb{R}^3$. The straight line $\mathbb{L}_{P_0,E}$ is described by the equation – system

$$\mathbb{L}_{P_0,E}: \quad \begin{cases} x_t = x_0 + te_x \\ y_t = y_0 + te_y \\ z_t = z_0 + te_z \end{cases} \quad , -\infty < t < \infty \tag{13}$$

too. Clearly, for the distance between P_0 and P_t we have $d(P_0, P_t) = |t|$.

Exploding the straight line $\mathbb{L}_{P_0,E}$ we have the super-line $\widetilde{\mathbb{L}_{P_0,E}}$ which has the equation-system

$$\widetilde{\mathbb{L}_{P_0,E}}: \begin{cases} \breve{x}_t = \breve{x}_0 \overline{\oplus} \left(\breve{t} \,\overline{\odot}\, \breve{e}_x\right) \\ \breve{y}_t = \breve{y}_0 \overline{\oplus} \left(\breve{t} \,\overline{\odot}\, \breve{e}_y\right) \\ \breve{z}_t = \breve{z}_0 \overline{\oplus} \left(\breve{t} \,\overline{\odot}\, \breve{e}_z\right) \end{cases} , -\infty < t < \infty \tag{14}$$

where (5) (in the case $\sigma = 1$) and the super–operations given by Postulates of super–addition and super–multiplication were used. Of course, $\widetilde{\mathbb{L}_{P_0,E}} \subset \widetilde{\mathbb{R}^3}$. Finally, the *extra-line* is defined by

$$\mathbb{L}_{P_0,E}^{extra} = \left(\widetilde{\mathbb{L}_{P_0,E}}\right)_{box} \tag{15}$$

see (8).

In other words the extra-line is the visible open super-passage of the super-line $\widetilde{\mathbb{L}_{P_0,E}}$. Its endpoints are invisible and situated on the invisible border of the universe $\mathbb{R}^3$. The coordinates of endpoints are determined by the discriminators $\breve{-1}$ and $\breve{1}$. (If the parameter σ is „big enough" the extra-line may be a good approximation for the way of the light in the case of large distances.)

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In the next we need the concepts of super-subtraction and super–division

$$\breve{x}\ \overline{\ominus}\ \breve{y} = \breve{x-y} \quad , x, y \in \mathbb{R} \tag{16}$$

and

$$\breve{x}\ \overline{\oslash}\ \breve{y} = \breve{\left(\frac{x}{y}\right)} \quad , x, y(\neq 0) \in \mathbb{R} \tag{17}$$

respectively.

To find the coordinates of endpoint we are looking for its parameters t_{small} and t_{big} , among the solutions of the following six equations

$$\breve{x}_0 \overline{\oplus} (\breve{t}\, \overline{\odot}\, \breve{e}_x) = \breve{1} \text{ solution } t = \underline{(\breve{1}\, \overline{\ominus}\, \breve{x}_0)\, \overline{\oslash}\ \breve{e}_x}, \text{where } \breve{e}_x \neq 0 \tag{18}$$

$$\breve{x}_0 \overline{\oplus} (\breve{t}\, \overline{\odot}\, \breve{e}_x) = \breve{-1} \text{ solution } t = \underline{(-\breve{1}\, \overline{\ominus}\, \breve{x}_0)\, \overline{\oslash}\ \breve{e}_x}, \text{where } \breve{e}_x \neq 0 \tag{19}$$

$$\breve{y}_0 \overline{\oplus} (\breve{t}\, \overline{\odot}\, \breve{e}_y) = \breve{1} \text{ solution } t = \underline{(\breve{1}\, \overline{\ominus}\, \breve{y}_0)\, \overline{\oslash}\ \breve{e}_y}, \text{where } \breve{e}_y \neq 0 \tag{20}$$

$$\breve{y}_0 \overline{\oplus} (\breve{t}\, \overline{\odot}\, \breve{e}_y) = \breve{-1} \text{ solution } t = \underline{(\breve{-1}\, \overline{\ominus}\, \breve{y}_0)\, \overline{\oslash}\ \breve{e}_y}, \text{where } \breve{e}_y \neq 0 \tag{21}$$

$$\breve{z}_0 \overline{\oplus} (\breve{t}\, \overline{\odot}\, \breve{e}_z) = \breve{1} \text{ solution } t = \underline{(\breve{1}\, \overline{\ominus}\, \breve{z}_0)\, \overline{\oslash}\ \breve{e}_z}, \text{where } \breve{e}_z \neq 0 \tag{22}$$

$$\breve{z}_0 \overline{\oplus} (\breve{t}\, \overline{\odot}\, \breve{e}_z) = \breve{-1} \text{ solution } t = \underline{(-\breve{1}\, \overline{\ominus}\, \breve{z}_0)\, \overline{\oslash}\ \breve{e}_z}, \text{where } \breve{e}_z \neq 0 \tag{23}$$

If $\breve{e}_x = 0$, then (18) and (19), if $\breve{e}_y = 0$, then (20) and (21), $\breve{e}_z = 0$, then (22) and (23) are omitted. If $t_{small} < t < t_{big}$ then we have

$$\breve{-1} < \breve{x}_0 \overline{\oplus} (\breve{t}\, \overline{\odot}\, \breve{e}_x) < \breve{1},$$
$$\breve{-1} < \breve{y}_0 \overline{\oplus} (\breve{t}\, \overline{\odot}\, \breve{e}_y) < \breve{1},$$
$$\breve{-1} < \breve{z}_0 \overline{\oplus} (\breve{t}\, \overline{\odot}\, \breve{e}_z) < \breve{1}.$$

Hence, using (9),(10) and the super–operations given by Postulates of super–addition and super– multiplication we have the equation–system of extra-line

$$\mathbb{L}_{P_0,E}^{extra} = \left\{ (x, y, z) \in \mathbb{R}^3 \middle| \begin{cases} x = \tanh^{-1}(x_0 + te_x) \\ y = \tanh^{-1}(y_0 + te_y) \\ z = \tanh^{-1}(z_0 + te_z) \end{cases}, (-\infty <)\, t_{small} < t < t_{big} (< \infty) \right\}$$

As the mentioned method seems to be complicated we give an example: Let be

$$P_0 = \left(\tanh\frac{1}{2}, \tanh 1, \tanh\frac{3}{2}\right) \in \underline{\mathbb{R}^3}$$

and

$$E = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right) \in \mathbb{R}^3$$

Repeating (12) , the equation – system (13) has the form

$$\mathbb{L}_{P_0,E}: \begin{cases} x_t = \tanh\frac{1}{2} + \frac{t}{\sqrt{6}} \\ y_t = \tanh 1 + \frac{t}{\sqrt{6}} \\ z_t = \tanh\frac{3}{2} - \frac{2t}{\sqrt{6}} \end{cases}, -\infty < t < \infty$$

Using the second inversion formula in Postulate of extension (14) will

$$\widetilde{\mathbb{L}_{P_0,E}}: \begin{cases} \breve{x}_t = \frac{1}{2} \overline{\oplus} \left(\breve{t}\, \overline{\odot}\, \breve{\left(\frac{1}{\sqrt{6}}\right)}\right) \\ \breve{y}_t = 1 \overline{\oplus} \left(\breve{t}\, \overline{\odot}\, \breve{\left(\frac{1}{\sqrt{6}}\right)}\right) \\ \breve{z}_t = \frac{3}{2} \overline{\oplus} \left(\breve{t}\, \overline{\odot}\, \breve{\left(\frac{-2}{\sqrt{6}}\right)}\right) \end{cases}, -\infty < t < \infty$$

The solutions of equations (18) – (23) are

$$t = \underline{(\breve{1}\, \overline{\ominus}\, \breve{x}_0)\, \overline{\oslash}\ \breve{e}_x} = \left(1 - \tanh\frac{1}{2}\right) \cdot \sqrt{6} \approx 1.317538506$$
$$t = \underline{(-\breve{1}\, \overline{\ominus}\, \breve{x}_0)\, \overline{\oslash}\ \breve{e}_x} = -\left(1 + \tanh\frac{1}{2}\right) \cdot \sqrt{6} \approx -3.581440979$$
$$t = \underline{(\breve{1}\, \overline{\ominus}\, \breve{y}_0)\, \overline{\oslash}\ \breve{e}_y} = (1 - \tanh 1) \cdot \sqrt{6} \approx 0.5839726696$$
$$t = \underline{(\breve{-1}\, \overline{\ominus}\, \breve{y}_0)\, \overline{\oslash}\ \breve{e}_y} = -(1 + \tanh 1) \cdot \sqrt{6} \approx -4.315006816,$$
$$t = \underline{(\breve{1}\, \overline{\ominus}\, \breve{z}_0)\, \overline{\oslash}\ \breve{e}_z} = -\left(1 - \tanh\frac{3}{2}\right) \cdot \sqrt{\frac{3}{2}} \approx -0.1161691899,$$
$$t = \underline{(-\breve{1}\, \overline{\ominus}\, \breve{z}_0)\, \overline{\oslash}\ \breve{e}_z} = \left(1 + \tanh\frac{3}{2}\right) \cdot \sqrt{\frac{3}{2}} \approx 2.33320553.$$

Hence, $t_{small} = -\left(1 - \tanh\frac{3}{2}\right) \cdot \sqrt{\frac{3}{2}}$ and $t_{big} = (1 - \tanh 1) \cdot \sqrt{6}$. Finally,

$$\mathbb{L}_{P_0,E}^{extra} = \left\{ (x, y, z) \in \mathbb{R}^3 \middle| \begin{cases} x = \tanh^{-1}\left(\tanh\frac{1}{2} + \frac{t}{\sqrt{6}}\right) \\ y = \tanh^{-1}\left(\tanh 1 + \frac{t}{\sqrt{6}}\right) \\ z = \tanh^{-1}\left(\tanh\frac{3}{2} - \frac{2t}{\sqrt{6}}\right) \end{cases}, t_{small} < t < t_{big} \right\} \tag{24}$$

We can obseve that

$$\lim_{\substack{t \to t_{small} \\ t_{small} < t}} \tanh^{-1}\left(\tanh\frac{3}{2} - \frac{2t}{\sqrt{6}}\right) = \infty$$

and

$$\lim_{\substack{t \to t_{big} \\ t < t_{big}}} \tanh^{-1}\left(\tanh 1 + \frac{t}{\sqrt{6}}\right) = \infty,$$

which refer to the invisible endpoints

$$\left(\tanh^{-1}\left(\tanh\frac{1}{2} + \frac{1}{2}\tanh\frac{3}{2} - \frac{1}{2}\right), \tanh^{-1}\left(\tanh 1 + \frac{1}{2}\tanh\frac{3}{2} - \frac{1}{2}\right), \breve{1}\right) \tag{25}$$

and

$$\left(\tan^{-1}\left(1 + \tanh\frac{1}{2} - \tanh 1\right), \breve{1}, \tanh^{-1}\left(\tanh\frac{3}{2} + 2\tanh 1 - 2\right)\right) \tag{26}$$

We remark, that there is no euclidean plane containing the

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extra-line $\mathbb{L}^{extra}_{P_0,E}$ under (24).

Here we mention some properties of extra geometry concerning extra-lines.

Property 1. If $\mathcal{P}_1 = (u_1, v_1, w_1)$ and $\mathcal{P}_2 = (u_2, v_2, w_2)$ are given different points in the three– dimensional space $\mathbb{R}^3$ then there exists one and only one extra-line which contains them.

Proof. By the Postulate of extension the points $\underline{\mathcal{P}_1} = \left(\underline{u_1}, \underline{v_1}, \underline{w_1}\right)$ and $\underline{\mathcal{P}_2} = \left(\underline{u_2}, \underline{v_2}, \underline{w_2}\right) \in \underline{\mathbb{R}}^3$ are mutually and unambiguously determined. Choosing $P_0 = \underline{\mathcal{P}_1}$ and $E = \frac{1}{\left\|\underline{\mathcal{P}_2}-\underline{\mathcal{P}_1}\right\|} \cdot \left(\underline{\mathcal{P}_2} - \underline{\mathcal{P}_1}\right)$, the method via (12) – (15) yields:

$$\mathbb{L}^{extra}_{(\mathcal{P}_1,\mathcal{P}_2)} = \left\{(x,y,z) \in \mathbb{R}^3 \middle| \begin{cases} x = \tanh^{-1}\left(\underline{u_1} + t \cdot \frac{\underline{u_2}-\underline{u_1}}{\left\|\underline{\mathcal{P}_2}-\underline{\mathcal{P}_1}\right\|}\right) \\ y = \tanh^{-1}\left(\underline{v_1} + t \cdot \frac{\underline{v_2}-\underline{v_1}}{\left\|\underline{\mathcal{P}_2}-\underline{\mathcal{P}_1}\right\|}\right) \\ z = \tanh^{-1}\left(\underline{w_1} + t \cdot \frac{\underline{w_2}-\underline{w_1}}{\left\|\underline{\mathcal{P}_2}-\underline{\mathcal{P}_1}\right\|}\right) \end{cases}, t_{small} < t < t_{big} \right\} \quad (27)$$

If $t = 0$ the (27) yields the point $\mathcal{P}_1$ and $t = \left\|\underline{\mathcal{P}_2} - \underline{\mathcal{P}_1}\right\|$ gives the point $\mathcal{P}_2$. Of course, we can exchange points $\mathcal{P}_1$ and $\mathcal{P}_2$.

Example 1: Let be

$\mathcal{P}_1 = \left(\frac{1}{2}, ,1, \frac{3}{2}\right)$ and $\mathcal{P}_2 = \left(\tanh^{-1}\left(\tanh\frac{1}{2} + \frac{1}{2\sqrt{6}}\right), \tanh^{-1}\left(\tanh 1 + \frac{1}{2\sqrt{6}}\right), \tanh^{-1}\left(\tanh\frac{3}{2} - \frac{1}{\sqrt{6}}\right)\right)$.

Now $\underline{u_1} = \tanh\frac{1}{2}$, $\underline{v_1} = \tanh 1$, $\underline{w_1} = \tanh\frac{3}{2}$, $\underline{u_2} - \underline{u_1} = \frac{1}{2\sqrt{6}}$, $\underline{v_2} - \underline{v_1} = \frac{1}{2\sqrt{6}}$, $\underline{w_2} - \underline{w_1} = -\frac{1}{\sqrt{6}}$ and $\left\|\underline{\mathcal{P}_2} - \underline{\mathcal{P}_1}\right\| = \frac{1}{2}$. So, (27) yields the equation-system under (24), that is $\mathbb{L}^{extra}_{P_0,E} = \mathbb{L}^{extra}_{(\mathcal{P}_1,\mathcal{P}_2)}$. To control our computation we mention that for the parameter $t = \frac{1}{2}\left(< t_{big} = (1 - \tanh 1) \cdot \sqrt{6}\right)$ by (24) we have $\mathcal{P}_2$.

Property 2. If $\mathcal{P}_1 = (u_1, v_1, w_1) \in \mathbb{R}^3$ and $\mathcal{P}_2 = (u_2, v_2, w_2)$ is situated in the (invisible) border of the three– dimensional space $\mathbb{R}^3$ then there exists one and only one extra-line which contains the point $\mathcal{P}_1$ and one of its invisible endpoint is $\mathcal{P}_2$.

Proof. By the Postulate of extension the points $\underline{\mathcal{P}_1} = \left(\underline{u_1}, \underline{v_1}, \underline{w_1}\right)$ and $\underline{\mathcal{P}_2} = \left(\underline{u_2}, \underline{v_2}, \underline{w_2}\right) \in \underline{\mathbb{R}}^3$ are mutually and unambiguously determined. Now, $\underline{\mathcal{P}_1}$ is inside of open cube $\underline{\mathbb{R}}^3$ (see (11)) and $\underline{\mathcal{P}_2}$ is its border. The other parts of our proof is similar to the proof of Property 1, so we omit it. We can use (27), again.

Example 1:* Let be

$\mathcal{P}_1 = \left(\frac{1}{2}, ,1, \frac{3}{2}\right)$ and $\mathcal{P}_2$ is given by (25). Now $\underline{u_1} = \tanh\frac{1}{2}$, $\underline{v_1} = \tanh 1$, $\underline{w_1} = \tanh\frac{3}{2}$, $\underline{u_2} - \underline{u_1} = \frac{1}{2}\tanh\frac{3}{2} - \frac{1}{2}$, $\underline{v_2} - \underline{v_1} = \frac{1}{2}\tanh\frac{3}{2} - \frac{1}{2}$, $\underline{w_2} - \underline{w_1} = 1 - \tanh\frac{3}{2}$ and $\left\|\underline{\mathcal{P}_2} - \underline{\mathcal{P}_1}\right\| = \left(1 - \tanh\frac{3}{2}\right)\sqrt{\frac{3}{2}}$. So, by (27) we get:

$$\mathbb{L}^{extra}_{(\mathcal{P}_1,\mathcal{P}_2)} = \left\{(x,y,z) \in \mathbb{R}^3 \middle| \begin{cases} x = \tanh^{-1}\left(\tanh\frac{1}{2} + t \cdot \frac{-1}{\sqrt{6}}\right) \\ y = \tanh^{-1}\left(\tanh 1 + t \cdot \frac{-1}{\sqrt{6}}\right) \\ z = \tanh^{-1}\left(\tanh\frac{3}{2} + t \cdot \frac{2}{\sqrt{6}}\right) \end{cases}, -(1 - \tanh 1) \cdot \sqrt{6} < t < \left(1 - \tanh\frac{3}{2}\right) \cdot \sqrt{\frac{3}{2}} \right\} \quad (28)$$

If $t = 0$ then (28) gives the point $\mathcal{P}_1$ while

$$\lim_{t \to \left(1-\tanh\frac{3}{2}\right)\cdot\sqrt{\frac{3}{2}}} \tanh^{-1}\left(\tanh\frac{1}{2} + t \cdot \frac{-1}{\sqrt{6}}\right) = \tanh^{-1}\left(\tanh\frac{1}{2} + \frac{1}{2}\tanh\frac{3}{2} - \frac{1}{2}\right),$$

$$\lim_{t \to \left(1-\tanh\frac{3}{2}\right)\cdot\sqrt{\frac{3}{2}}} \tanh^{-1}\left(\tanh 1 + t \cdot \frac{-1}{\sqrt{6}}\right) = \tanh^{-1}\left(\tanh 1 + \frac{1}{2}\tanh\frac{3}{2} - \frac{1}{2}\right)$$

and

$$\lim_{t \to \left(1-\tanh\frac{3}{2}\right)\cdot\sqrt{\frac{3}{2}}} \tanh^{-1}\left(\tanh\frac{3}{2} + t \cdot \frac{2}{\sqrt{6}}\right) = \infty$$

refer to the invisible endpoint $\mathcal{P}_2$.

We remark, that

$$\lim_{t \to -(1-\tanh 1)\cdot\sqrt{6}} \tanh^{-1}\left(\tanh\frac{1}{2} + t \cdot \frac{-1}{\sqrt{6}}\right) = \tanh^{-1}\left(1 + \tanh\frac{1}{2} - \tanh 1\right),$$

$$\lim_{t \to -(1-\tanh 1)\cdot\sqrt{6}} \tanh^{-1}\left(\tanh 1 + t \cdot \frac{-1}{\sqrt{6}}\right) = \infty$$

and

$$\lim_{t \to -(1-\tanh 1)\cdot\sqrt{6}} \tanh^{-1}\left(\tanh\frac{3}{2} + t \cdot \frac{2}{\sqrt{6}}\right) = \tanh^{-1}\left(\tanh\frac{3}{2} - 2\tanh 1 - 2\right)$$

refer to the other invisible endpoint of $\mathbb{L}^{extra}_{(\mathcal{P}_1,\mathcal{P}_2)}$ which was already mentioned under (26).

Property 3. If the different points $\mathcal{P}_1 = (u_1, v_1, w_1)$ and $\mathcal{P}_2 = (u_2, v_2, w_2)$ are situated in the (invisible) border of the three – dimensional space $\mathbb{R}^3$such that their compressed points $\underline{\mathcal{P}_1}$ and $\underline{\mathcal{P}_2}$ there are no the same side of the closed cube

$$closed\ \underline{\mathbb{R}}^3 = \left\{(x,y,z) \in \mathbb{R}^3 \middle| \begin{cases} -1 \le x \le 1 \\ -1 \le y \le 1 \\ -1 \le z \le 1 \end{cases} \right\}$$

then there exists one and only one extra-line which has the endpoints $\mathcal{P}_1$and $\mathcal{P}_2$.

Proof. By the Postulate of extension the points $\underline{\mathcal{P}_1} = \left(\underline{u_1}, \underline{v_1}, \underline{w_1}\right)$ and $\underline{\mathcal{P}_2} = \left(\underline{u_2}, \underline{v_2}, \underline{w_2}\right) \in \underline{\mathbb{R}}^3$ are mutually and unambiguously determined. Choosing $P_0 = \frac{1}{2}\left(\underline{\mathcal{P}_1} + \underline{\mathcal{P}_2}\right)$ and $E = \frac{1}{\left\|\underline{\mathcal{P}_2}-\underline{\mathcal{P}_1}\right\|} \cdot \left(\underline{\mathcal{P}_2} - \underline{\mathcal{P}_1}\right)$, As $P_0 \in \underline{\mathbb{R}}^3$, the method via (12) – (15) yields

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$$\mathbb{L}^{extra}_{(\mathcal{P}_1,\mathcal{P}_2)} = \left\{ (x,y,z)\in\mathbb{R}^3 \middle| \begin{cases} x = \tanh^{-1}\left(\frac{\underline{u_1}+\underline{u_2}}{2} + t\cdot\frac{\underline{u_2}-\underline{u_1}}{\left\|\underline{\mathcal{P}_2}-\underline{\mathcal{P}_1}\right\|}\right) \\ y = \tanh^{-1}\left(\frac{\underline{v_1}+\underline{v_2}}{2} + t\cdot\frac{\underline{v_2}-\underline{v_1}}{\left\|\underline{\mathcal{P}_2}-\underline{\mathcal{P}_1}\right\|}\right) \\ z = \tanh^{-1}\left(\frac{\underline{w_1}+\underline{w_2}}{2} + t\cdot\frac{\underline{w_2}-\underline{w_1}}{\left\|\underline{\mathcal{P}_2}-\underline{\mathcal{P}_1}\right\|}\right) \end{cases}, \; -\frac{\left\|\underline{\mathcal{P}_2}-\underline{\mathcal{P}_1}\right\|}{2} < t < \frac{\left\|\underline{\mathcal{P}_2}-\underline{\mathcal{P}_1}\right\|}{2} \right\} \quad (29)$$

*Example 2**. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be given by (25) and (26), respectively. Now,

$\frac{\underline{u_1}+\underline{u_2}}{2} = \frac{1}{4} + \tanh\frac{1}{2} + \frac{1}{4}\tanh\frac{3}{2} - \frac{1}{2}\tanh 1 \approx 0.5576071427$;
$\frac{\underline{v_1}+\underline{v_2}}{2} = \frac{1}{4} + \frac{1}{4}\tanh\frac{3}{2} + \frac{1}{2}\tanh 1 \approx 0.8570841414$;
$\frac{\underline{w_1}+\underline{w_2}}{2} = -\frac{1}{2} + \frac{1}{2}\tanh\frac{3}{2} + \tanh 1 \approx 0.7141682828$;
$\underline{u_2} - \underline{u_1} = \frac{3}{2} - \frac{1}{2}\tanh\frac{3}{2} - \tanh 1 \approx 0.2858317172$;
$\underline{v_2} - \underline{v_1} = \frac{3}{2} - \frac{1}{2}\tanh\frac{3}{2} - \tanh 1 \approx 0.2858317172$;
$\underline{w_2} - \underline{w_1} = -3 + \tanh\frac{3}{2} + 2\tanh 1 \approx -0.5716634344$;
$\left\|\underline{\mathcal{P}_2} - \underline{\mathcal{P}_1}\right\| = 3\cdot\left(\sqrt{\frac{3}{2}} - \frac{1}{\sqrt{6}}\tanh\frac{3}{2} - \sqrt{\frac{2}{3}}\cdot\tanh 1\right) \approx 0.7001418594$.

So, by (29) the equation–system which describes the extra-line $\mathbb{L}^{extra}_{(\mathcal{P}_1,\mathcal{P}_2)}$ is

$$\mathbb{L}^{extra}_{(\mathcal{P}_1,\mathcal{P}_2)}: \begin{cases} x = \tanh^{-1}\left(\frac{1}{4} + \tanh\frac{1}{2} + \frac{1}{4}\tanh\frac{3}{2} - \frac{1}{2}\tanh 1 + t\cdot\frac{1}{\sqrt{6}}\right) \\ y = \tanh^{-1}\left(\frac{1}{4} + \frac{1}{4}\tanh\frac{3}{2} + \frac{1}{2}\tanh 1 + t\cdot\frac{1}{\sqrt{6}}\right) \\ z = \tanh^{-1}\left(-\frac{1}{2} + \frac{1}{2}\tanh\frac{3}{2} + \tanh 1 + t\cdot\frac{-2}{\sqrt{6}}\right) \end{cases}, \; -\frac{\left\|\underline{\mathcal{P}_2}-\underline{\mathcal{P}_1}\right\|}{2} < t < \frac{\left\|\underline{\mathcal{P}_2}-\underline{\mathcal{P}_1}\right\|}{2} \quad (30)$$

The equation–sytems under (30) and (24) represent the same extra-line. If in (30) we choose the parameter $t = \sqrt{6}\cdot\left(\frac{1}{2}\tanh 1 - \frac{1}{4}\tanh\frac{3}{2} - \frac{1}{4}\right) \approx -0.2339017399$, the point $\left(\frac{1}{2}, 1, \frac{3}{2}\right)$ is obtained.

III. The Extra Parallelism

Similarly to the Part II we use the $\sigma = 1$, again. Let $\mathbb{S}$ be an Euclidean plane such that $\mathbb{S} \cap \underline{\mathbb{R}^3} \neq \{\ \}$. Exploding $\mathbb{S}$ we have the super–plane $\breve{\mathbb{S}}$ (See, (5)). Clearly, $\breve{\mathbb{S}} \subset \widetilde{\mathbb{R}^3}$. The visible subset of super – plane is the *extra-plane* $\mathbb{S}^{extra}$. Namely,

$$\mathbb{S}^{extra} = \breve{\mathbb{S}}_{box}\text{ , (See, (8))}$$

The general discussion of the concept of extra-plane will be in a following article. Now, we establish that the "x, y" – coordinate plane of the three–dimensional space $\mathbb{R}^3$ is an extra-plane, too. (Of course, it is a Euclidean plane, but in general, the extra-plane is not Euclidean plane.) The x – axis of the Descartes coordinate–system is an extra-line with the endpoints $(\widetilde{-1}, 0, 0)$ and $(\breve{1}, 0, 0)$ on the invisible border of the three–dimensional space. Similarly, y – axis as an extra-line has the invisible endpoints $(0, \widetilde{-1}, 0)$ and $(0, \breve{1}, 0)$.

Definition 1. (The concept of extra parallelism.) If the extra-lines $\mathbb{L}_1^{extra}$ and $\mathbb{L}_2^{extra}$ have a joint endpoint in the border of the three–dimensional space $\mathbb{R}^3$ then they are called *extra parallels of each other*.

We remark that if both endpoint are joint, then $\mathbb{L}_1^{extra} = \mathbb{L}_2^{extra}$ (See, Property 3).

By the construction of extra-line and Definition 1 we mention an obvious fact as

Lemma 1. The extra-lines $\mathbb{L}_1^{extra}$ and $\mathbb{L}_2^{extra}$ are extra parallels of each other if and only if the straight lines $\mathbb{L}_1$ and $\mathbb{L}_2$ have a joint pont of the border of the open cube $\underline{\mathbb{R}^3}$. (See (11).)

Let us consider the extra-line $\mathbb{L}^{extra}_{(\mathcal{P}_1,\mathcal{P}_2)}$ with its invisible endpoints $\mathcal{P}_1 = (0, \breve{1}, 0)$ and $\mathcal{P}_2 = (\breve{1}, 0, 0)$. By (29) we have

$$\mathbb{L}^{extra}_{(\mathcal{P}_1,\mathcal{P}_2)} = \left\{ (x,y,z)\in\mathbb{R}^3 \middle| \begin{cases} x = \tanh^{-1}\left(\frac{1}{2} + t\cdot\frac{1}{\sqrt{2}}\right) \\ y = \tanh^{-1}\left(\frac{1}{2} - t\cdot\frac{1}{\sqrt{2}}\right) \\ z = 0 \end{cases}, -\frac{\sqrt{2}}{2} < t < \frac{\sqrt{2}}{2} \right\} \quad (31)$$

Hence, the explicit form $y = \tanh^{-1}(1 - \tanh x)$ where $(0 < x < \infty)$ is obtained.

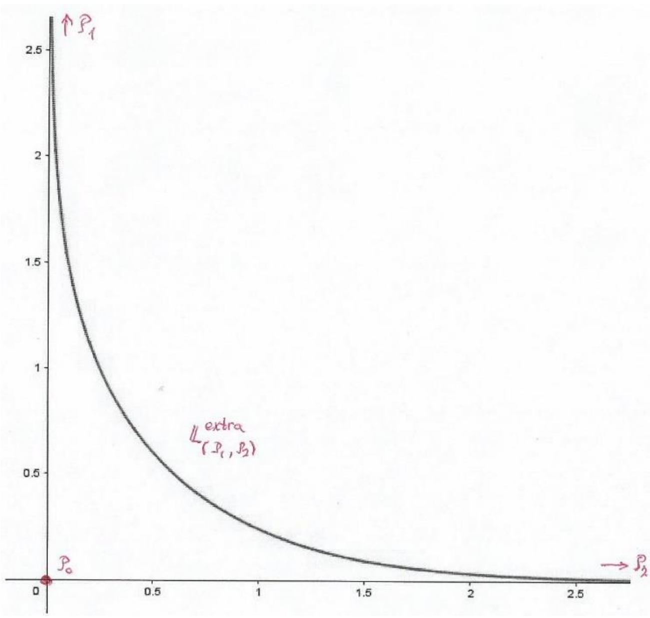

Fig. 3.

We observe, that the "y" – axis of the Descartes coordinate–system and $\mathbb{L}^{extra}_{(\mathcal{P}_1,\mathcal{P}_2)}$ described by (31) have the joint endpoint $\mathcal{P}_1 = (0, \breve{1}, 0)$. So, by Definition 1 they are extra parallels of each other. This fact is reflected on Fig. 3. („Appointment" in the „infinity".) Considering the endpoint $\mathcal{P}_2 = (\breve{1}, 0, 0)$ we get that the "x" – axis of the Descartes coordinate–system and $\mathbb{L}^{extra}_{(\mathcal{P}_1,\mathcal{P}_2)}$ are also extra parallels of each other. As the the $O = (0, 0, 0)$ is the point of intersection of axes we can see a very inportant property of extra geometry

Property 4. In the three–dimensional space $\mathbb{R}^3$, there exist an extra-plane $\mathbb{S}^{extra}$, a point $\mathcal{P}_0$, and an extra-line $\mathbb{L}^{extra}_{(\mathcal{P}_1,\mathcal{P}_2)}$

such that

$$\mathcal{P}_0 \notin \mathbb{L}^{extra}_{(\mathcal{P}_1,\mathcal{P}_2)} \tag{32}$$

and

$$\left(\mathcal{P}_0 \cup \mathbb{L}^{extra}_{(\mathcal{P}_1,\mathcal{P}_2)}\right) \subset \mathbb{S}^{extra} \tag{33}$$

then there exist two extra-lines $\mathbb{L}_1^{extra}$ and $\mathbb{L}_2^{extra}$ such that

$$(\mathbb{L}_1^{extra} \cup \mathbb{L}_2^{extra}) \subset \mathbb{S}^{extra}, \tag{34}$$

$$\mathcal{P}_0 \in (\mathbb{L}_1^{extra} \cap \mathbb{L}_2^{extra}) \tag{35}$$

and

$$\mathbb{L}_1^{extra} \neq \mathbb{L}_2^{extra} \tag{36}$$

are fulfilled.

Property 4 gives a similarity with Bolyai – Lobachewskian geometry (See [8]).

Property 5. The extra parallelism is not transitive.

Proof. Le tus consider the "x"and"y" axes of the Descartes coordinate – system. Both axes are extra parallel with $\mathbb{L}^{extra}_{(\mathcal{P}_1,\mathcal{P}_2)}$ described by (31) but they are not extra parallel each other (See, Fig 3).

Property 6. In the extra geometry there exist extra-lines $\mathbb{L}_1^{extra}$ and $\mathbb{L}_2^{extra}$ and extra-plane $\mathbb{S}^{extra}$ such that

$$(\mathbb{L}_1^{extra} \cup \mathbb{L}_2^{extra}) \subset \mathbb{S}^{extra}, \tag{37}$$

$$\mathbb{L}_1^{extra} \cap \mathbb{L}_2^{extra} = \{\ \ \} \tag{38}$$

and $\mathbb{L}_1^{extra}$ and $\mathbb{L}_2^{extra}$ are not extra parallels of each other.

Proof. Let us consider on the "x , y" coordinate plane (which is an extra plane, too) the Euclidean straight line $\mathbb{L}_1$ given by the equation:

$$y = \underline{1}(= \tanh 1 \approx 0.761594156).$$

It has two points of intersection with the border of open cube $\underline{\mathbb{R}^3}$. They are:

$$P_* = (-1\,,\underline{1}\,,0) \text{ and } P_{**} = (-1\,,\underline{1}\,,0).$$

Exploding the straight line $\mathbb{L}_1$we have the super–line $\overset{\smile}{\mathbb{L}_1}$ given by the equation $y = 1$. (If $x \leq \overset{\smile}{-1}$ or $x \geq \check{1}$ the point (x, y) is already invisible.) Hence, we have the extra-line $\mathbb{L}_1{}^{extra}$ given by the equation $y = 1$ with $-\infty < x < \infty$. Its endpoints are

$$\check{P}_* = (\overset{\smile}{-1}\,, 1, 0) \text{ and } \check{P}_{**} = (\check{1}\,, 1\,, 0).$$

In the next, let us consider the "x" – axis of the Descartes coordinate–system as $\mathbb{L}_2^{extra}$, having the endpoints

$$\check{P}_1 = (\overset{\smile}{-1}\,, 0\,, 0) \text{ and } \check{P}_2 = (\check{1}\,, 0\,, 0).$$

As the "x , y" coordinate plane is an extra plane, the statement (37) is fulfilled. Of course, we have (38). Finally, we can see, that $\mathbb{L}_1^{extra}$ and $\mathbb{L}_2^{extra}$ have no joint endpoints at all, so, by the Definition 1 they are not extra parallels of each other.

Properties 4, 5 and 6 show the deviation from Euclidean-geometry.

We remark, that the extra-lines $\mathbb{L}_1^{extra}$ and $\mathbb{L}_2^{extra}$ mentioned in the proof of Property 6 are parallel straight lines in Euclidean geometry but not in the extra geometry. In the general case instead of (31) we get

$$\mathbb{L}^{\sigma-extra}_{(\mathcal{P}_1,\mathcal{P}_2)} = \left\{ (x,y,z) \in \mathbb{R}^3 \middle| \begin{cases} x = \sigma \tanh^{-1}\left(\frac{1}{2} + t \cdot \frac{1}{\sigma\sqrt{2}}\right) \\ y = \sigma \tanh^{-1}\left(\frac{1}{2} - t \cdot \frac{1}{\sigma\sqrt{2}}\right), -\frac{\sigma\sqrt{2}}{2} < t < \frac{\sigma\sqrt{2}}{2} \\ z = 0 \end{cases} \right\}$$

but the Property 4 remains valid.

IV. Discussion

In the article [4] we modeled a Multiverse in which our Universe is just one of the many. The universes of the Multiverse can also be overlapping, which means that each point in the Multiverse is center of a universe. If two universes are closer to each other (see [4], part V.) they cannot be distinguished by today's technical means, but by calculations. The Euclidean geometry of Multiverse is also valid in our Universe but here the extra-line does not show the parallelism-property of Euclidean geometry. This article presents the parts of the super-lines of the Multiverse that fall into our Universe. The parts are usually curves in our Universe and called extra-lines. What is new that extra-lines are defined by their two points of intersection with the invisible boundary of our Universe.

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