Maximally Flat IIR Fullband Differentiators with Flat Group Delay Responses

Slaviša Ilić, Ahmad Mohammed Salih, Majid Hamid Abdullah, and Dragiša Milić

Abstract — A new design method for maximally flat IIR fullband differentiators with flat group delay response is derived in this paper. The design method starts from the flatness conditions of magnitude response and group delay response at the origin. After mathematical manipulations it shows that presented design method reduces to solving the system of linear equations. By increasing the orders of polynomials in numerator and denominator, degrees of flatness are increased, that is improvement in magnitude responses and group delay responses in terms of flatness is obtained.

Index Terms — fullband differentiators, maximally flat magnitude response, flat group delay response.

I. INTRODUCTION

Fullband differentiators are needed in various applications [1]-[7] where time-derivative of signal at input port need to be computed. Those filters, as any other type of filter functions, can be designed as finite impulse response filters (FIR) [8]-[15] and infinite impulse response filters (IIR) [16]-[23].

Frequency response of ideal fullband differentiators is as follows

\[ H_d(e^{j\omega}) = j\omega e^{-j\omega \tau_0}, \]  (1)

where \( \tau_0 \) is desired group delay response. In this paper, the method derived in [24] is extended to design of maximally flat IIR fullband differentiators with flat group delay responses, whose transfer function can be expressed as [25], [16].

\[ H(z) = (1 - z^{-1}) B(z) \frac{\bar{B}(z)}{A(z)}, \]  (2)

where

\[ A(z) = \sum_{i=0}^{N} a_i z^{-i}, \quad B(z) = \sum_{i=0}^{N} b_i z^{-i}, \]  (3)

\[ \bar{B}(z) = (1 - z^{-1}) B(z) = \sum_{i=0}^{M+1} b_i z^{-i}, \]  (4)

Let’s denote the phase response of \( H(z) \) as \( \theta(\omega) \). Obviously, \( \theta(\omega) = \pi/2 \) if \( B(z) \) does not have roots at \( z = 1 \). Besides Introduction, the paper consists of three sections entitled Design Method, Design Examples and Conclusion.

In section Design Method, a new design method is derived such that maximally flat IIR fullband differentiators with flat group delay responses are obtained. Design examples are then presented.

II. DESIGN METHOD

From equation (1) it follows that maximally flat IIR fullband differentiators with flat group delay responses are characterized by flatness conditions

\[ H( e^{j\omega} ) \big|_{\omega = 0} = 0, \]  (5)

\[ \frac{d^i |H(e^{j\omega})|}{d\omega^i} \bigg|_{\omega = 0} = 1, \]  (6)

\[ \frac{d^i |H(e^{j\omega})|}{d\omega^i} \bigg|_{\omega = 0} = 0, i = 2, 3, ..., K - 1, \]  (7)

\[ \frac{d^i \theta(\omega)}{d\omega^i} \bigg|_{\omega = 0} = -\tau_0, \]  (8)

\[ \frac{d^i \theta(\omega)}{d\omega^i} \bigg|_{\omega = 0} = 0, i = 2, 3, ..., K - 1, \]  (9)

where \( K \) is degree of flatness.

Applying Leibnitz’s rule for higher derivatives [26] of \( H(e^{j\omega}) = |H(e^{j\omega})| e^{j\theta(\omega)} \) at \( \omega = 0 \), and using previous equations, it shows that maximally flat IIR fullband differentiators with flat group delay responses have frequency responses satisfying

\[ \frac{d^i |H(e^{j\omega})|}{d\omega^i} \bigg|_{\omega = 0} = \sum_{k=0}^{i} \binom{i}{k} \frac{d^k |H(e^{j\omega})|}{d\omega^k} e^{j\theta(\omega)} \bigg|_{\omega = 0} = n, \quad \frac{d^{n+1} \theta(\omega)}{d\omega^{n+1}} \bigg|_{\omega = 0} = -j^n n \tau_0^{n-1}, \]  (10)

As from equation (2)

\[ H( e^{j\omega} ) A(e^{j\omega}) = \bar{B}(e^{j\omega}), \]  (11)

Differentiating \( n \) times and applying Leibnitz’s rule for higher derivatives [26] of left-hand side of previous equations, we get

\[ \frac{d^n}{d\omega^n} [H( e^{j\omega} ) A(e^{j\omega})] = \sum_{k=0}^{n} \binom{n}{k} \frac{d^k H(e^{j\omega})}{d\omega^k} \frac{d^{n-k} A(e^{j\omega})}{d\omega^{n-k}} = \frac{d^n}{d\omega^n} \bar{B}(e^{j\omega}), \]  (12)

and for \( \omega = 0 \) after substituting (10) following is obtained...

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Slaviša Ilić, Singidunum University, Serbia.
Ahmad Mohammed Salih, Higher Institute of Telecommunications, Iraq.
Majid Hamid Abdullah, Noontech Company, Iraq.
Dragiša Milić, Singidunum University, Serbia.
(e-mail: ddmilic@outlook.com)

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\[ \sum_{k=0}^{n}j^{-k}k \left( \frac{d^{n-k}A(e^{j\omega})}{d\omega^{n-k}} \right)_{\omega=0} = \frac{d^n\theta(e^{j\omega})}{d\omega^n} \bigg|_{\omega=0} \]  \hspace{1cm} (13)

Now, as from (3) and (4)
\[ d^{n-k}A(e^{j\omega}) \bigg|_{\omega=0} = (-j)^{n-k}\sum_{m=0}^{N}m^{n-k}a_m \]  \hspace{1cm} (14)
\[ d^n\theta(e^{j\omega}) \bigg|_{\omega=0} = (-j)^{n}\sum_{m=0}^{M+1}m^{n}b_m = -(-j)^{n}\sum_{m=0}^{M}(m + 1)^n - m^n)b_m, \]  \hspace{1cm} (15)

Then, equation (13) can be rewritten as
\[ \sum_{m=0}^{N}\sum_{k=1}^{n}k \left( \frac{\tau_0^{k-1}m^{n-k}a_m}{\sum_{m=0}^{M}} \right) = \sum_{m=0}^{M}((m + 1)^n - m^n)b_m. \]  \hspace{1cm} (16)

Since it can be adopted that \( a_0 = 1 \), the number of unknown coefficients is \( N + M + 1 \), and degree of flatness is also \( K = N + M + 1 \). In other words, (16) is satisfied for \( n = 1, 2, ..., K \). In matrix notation this reads
\[ [\mathbf{C} \mathbf{D}] \left[ a_1 a_2 ... a_n b_0 b_1 ... b_M \right]^T = \mathbf{f}, \]  \hspace{1cm} (17)
where \( \mathbf{C} \) is \( K \times N \) matrix, \( \mathbf{D} \) is \( K \times (M + 1) \) matrix and \( \mathbf{f} \) is \( K \times 1 \) vector:
\[ c_{nm} = \sum_{k=1}^{n}k \left( \frac{\tau_0^{k-1}m^{n-k}a_m}{\sum_{m=0}^{M}} \right), d_{nm} = (m - 1)^n - m^n, f_n = -\tau_0^{n-1}. \]  \hspace{1cm} (18)

Unknown coefficients are:
\[ [a_1 a_2 ... a_n b_0 b_1 ... b_M]^T = [\mathbf{C} \mathbf{D}]^{-1}\mathbf{f}. \]  \hspace{1cm} (19)

### III. DESIGN EXAMPLES

Two examples are presented in this section. **Example 1:** \( N = M = 3, \tau_0 \in \{2.4, 2.5\} \). Ideal and obtained magnitude responses and obtained group delay responses are given in Fig. 1.

**Example 2:** \( N = 3, M = 5, \tau_0 \in \{2.25, 2.5\} \). Ideal and obtained magnitude responses and obtained group delay responses are given in Fig. 2.

As can be concluded, proposed design method can be used to design IIR fullband differentiators. Determination of \( \tau_0 \) to obtain the best possible IIR fullband differentiator for some fixed value of \( N \) and \( M \) remains still an open problem.

### IV. CONCLUSION

A new IIR fullband differentiators design method is proposed in this paper. Resulting filters have maximally flat magnitude responses and flat group delay responses. By increasing the orders of polynomials in numerator and denominator, degrees of flatness are increased, i.e., there will be improvement in magnitude responses and group delay responses in terms of flatness. Proposed IIR fullband differentiators design method reduces to solving a system of linear equations. Determination of \( \tau_0 \) to obtain the best possible solution for some \( N \) and \( M \) remains still an open issue.

### REFERENCES


