Oscillations of a Dissipative Mechanical System Consisting of Solids

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Abstract—The paper deals with self-induced and forced linear oscillations, dissipative mechanical systems consisting of spatial bodies. The problem is solved by the Mueller method, the Gauss method, and the methods of theoretical mechanics. When solving the problems of intrinsic and forced oscillations of dissipatively inhomogeneous mechanical, new regularities of energy dissipation of mechanical systems are discovered.

Index Terms—Global Coefficient; Damping Coefficient; Frequency; Oscillations.

I. INTRODUCTION

The development of approaches to the development of methods and means of vibration protection leads to an expansion of the notion of a set of typical elements, which is associated with the detailed consideration of the dynamic properties of vibration protection systems that include more complex structures in the form of devices for converting motion and mechanisms. Analysis of methods and means of vibration protection and vibration isolation of technical objects shows that the majority of technical implementations of elastic and dissipative elements use mechanisms that bring dynamic features even to the simplest tasks of vibration protection. The development of generalized representations of vibration protection systems of various design and technical design allows us to formulate a demonstrative basis for expanding the set of typical elements and links. It is assumed that there are no external influences. For a quantitative sketch of the dissipative properties of the system as a whole, two values are proposed: the minimum decay rate of natural oscillations and the maximum resonant amplitude. The dissipative properties of the system are determined primarily by the damping characteristics of the systems, completely inapplicable to dissipative-inhomogeneous systems [1],[2].

It is established that the global damping characteristics of a dissipative-inhomogeneous system as a whole are determined not only (and not so much) by the viscoelastic properties of the elements of the system, as by the interaction of oscillations of various Eigen modes, which are substantially determined by structure, construction, geometry, size, the presence of elastic bonds, the arrangement of the elements of the system as a whole.

II. STATEMENT OF THE PROBLEM AND THE MAIN RELATIONS

Let us consider a mechanical system consisting of \( N \) absolutely rigid bodies connected to each other and to the base by \( L \) viscoelastic supports (Fig. I). The number of solids, their relative position, number, location and dynamic characteristics of viscoelastic elements can be arbitrary. Along with the fixed coordinate system \( O\xi\eta\zeta \), we introduce a coordinate system \( \vec{O}_{j}\vec{x}_{j}\vec{y}_{j}\vec{z}_{j} \) rigidly connected with the \( j \)-th and \( k \)-th bodies (Figure 1). We introduce the vector of generalized coordinates \( q \) of the system as a direct sum of vectors \( q_{j} \) for \( j = 1,2,\ldots,N \).

Denote sets of integers defining the numbers of viscoelastic supports attached to the \( j \)-th body through \( N_{j} \) \( (j = 1,2,\ldots,N) \), and \( N_{j} = \bigcup_{s=1}^{N} N_{s} \). Suppose further that the \( s \)-th support is arbitrarily oriented in space; the characteristics of such a support will be described by a matrix in the main axes of rigidity of the \( s \)-th support \( \overline{C}^{(s)} \); the mechanical characteristics of the \( s \)-th support are described by assumption by the linear hereditary Boltzmann-Voltaire theory [1]:

\[
F_{s}^{(s)}(t) = C^{(s)} \left[ \dot{x}^{(s)}(t) - \int_{0}^{t} \dot{\gamma}^{(s)}(t-\tau)x^{(s)}(\tau) d\tau \right]
\]

where \( x^{(s)} \) - the relative displacement of the ends of the \( s \)-th shock absorber;

Fig. 1. Dissipative mechanical system consisting of solids
The efforts arising in it; $F_i^{(s)}$ - operational rigidity; $C_i^{(s)}$ - instant stiffness; $R_i^{(s)}$ - the core of relaxation. It is assumed that some of the $L$ shock absorbers can be elastic; in this case the relaxation kernel is zero, and the operator rigidity coincides with the instantaneous. By viscoelastic supports here are meant shock absorbers made of polymeric materials with developed internal friction [2]. To describe the rheological behavior of such shock absorbers, the Sorokin model [3] or the hereditary Boltzmann-Voltaire theory [4],[5] is used. The calculation of the vibration system with a finite number of degrees of freedom is given in [4],[6] for the case when the rheological properties of all the shock absorbers of the system are identical. In this case, the motion of the system is a superposition of independent oscillations of normal coordinates. In the case of shock absorbers with different rheological characteristics, the situation changes. The interaction of oscillations of various normal coordinates occurs due to internal friction. It is natural to expect that this interaction will lead to new qualitative physical effects. The present work is devoted to studying these effects.

We introduce the matrix $S_{kp}^{(k)}$ (6x6) of the force transformation given in the $k$-th frame of reference at the point $P$ when it is brought to the point in the $j$-th frame of reference. The transformation matrix $S_{kp}^{(k)}$ is constructed from [6]. Equations of oscillations of mechanical systems have the form

$$[A][q] + [\tilde{C}][q] = \{Q(t)\}$$

where $\{q(t)\}$ - vector - column of generalized coordinates; $\{Q(t)\}$ - vector - the column of generalized forces; $[\tilde{C}]$-matrix 6Nx6N, whose elements are Voltaire operators of the form

$$\tilde{C}_{kl} = C_{kl} + \tilde{B}_{kl},$$

where $C_{kl}$-matrix of instantaneous generalized rigidities, $\tilde{B}_{kl}$-are elements of a non-negative definite matrix of generalized relaxation kernels, $\varphi$ -is an arbitrary function of time; $[A]$-matrix 6Nx6N, which has a block-diagonal structure

$$[A] = \begin{bmatrix} [A_1] & \cdots & [A_N] \end{bmatrix}$$

$[A_j]$ matrix of inertial coefficients of 6x6 dimension of a single $j$-th body; if the principal axes of inertia of the $j$-th body coincide with the axes of the fixed coordinate system $Ox\alpha \xi \eta \zeta$, then the matrix will be diagonal. In the case of a homogeneous system (all elements are made of a single viscoelastic material and described by identical relaxation nuclei, but different instantaneous stiness due to different sizes), all the relaxation nuclei in (2) are the same, $R_{kl} = R$. In the case of an inhomogeneous system (its deformable elements have different mechanical characteristics, in particular, some of them may be elastic), the situation changes. The operator coefficients $\tilde{C}_{kl}$ in (2) are now elements of the sum of two matrixes-the numerical and operator

$$\tilde{B}_{kl} = C_{kl} \int_0^1 R_{kl}(t - \tau)\varphi(\tau)d\tau$$

The paper investigates the intrinsic and forced oscillations of the system.

III. OWN OSCILLATIONS OF MECHANICAL SYSTEMS

In the problem of natural oscillations there are no generalized forces, the law of motion is sought in the form [7]

$$\{q\} = \{q_0\}e^{-i\omega t}$$

where $\omega = \omega_0 + i\omega_1$. The desired complex natural frequency, $\{q_0\}$ is the desired complex Eigen mode of oscillations.

The hereditary terms in (2) are assumed to be small in comparison with the elastic ones, whence the smallness of
the imaginary part of the natural frequency \( \omega_j \) as compared with the real part \( \omega_k \) follows. Using freezing method [7], equations (1) are written in the form:

\[
[A][q] + [C][q] = 0
\]

where \([C]\) is a complex matrix with elements

\[
\overline{C}_{kl} = C_{kl} \left[ 1 - \Gamma_{kl}^0(\omega_R) - i \Gamma_{kl}^0(\omega_R) \right]
\]

(3)

\[
\Gamma_{kl}^0(\omega_R) = \int_0^\infty R_{kl}(\tau) \cos \omega_R \tau d\tau , \quad \Gamma_{kl}^0(\omega_R) = \int_0^\infty R_{kl}(\tau) \sin \omega_R \tau d\tau ,
\]

\( R_{jk} \) - Matrix of generalized relaxation kernels, \( \omega_R \) - the real part of the required frequency, the Eigen frequency \( \omega \) is determined from the transcendental equation.

\[
[C - \omega^2 B] = 0
\]

(4)

where, \( C \) is a square stiffness matrix whose elements consist of the expressions (3), \( B \) is the square matrix of masses.

The roots of the characteristic equation are sought by the Muller method, the value of the left-hand side of (3) at each iteration of the Muller method is determined by the Gauss method with the separation of the principal element. Thus, the solution of equation (3) does not require the expansion of the determinant on the left-hand side. As the initial approximation, the natural frequencies of oscillations of the elastic system are chosen. The dependence of the Eigen frequencies and damping coefficients on the instantaneous rigidity \( C_{kl} \) at fixed values \( a_{kl} \) is investigated, and the relaxation nucleus \([8],[9]\).

\[
R_{kl} = A_{kl} \exp(-\beta t) / t^{1-\alpha}
\]

Fig. 3. Change the real and imaginary parts of the complex frequencies, depending on the stiffness factor. Dissipative non-homogeneous system.

As an example, we consider a system consisting of two absolutely rigid bodies connected to each other and a base by eight viscoelastic elements (Fig.2). As generalized coordinates, the center of mass and the rotation of the bodies relative to the axes of the inertial system are taken. The following parameter values are accepted: \( A = 0.048 = 0.05, \quad C_j = 1, \quad (j=1,\ldots,4), c_j = 0.1 \quad (j = 5,\ldots,7), \)

\( m_1 = m_2 = 7 \quad \text{The size of the object} \quad A \times B \times C = 0.42 \times 0.12 \times 0.28, \quad \text{fixing points} \quad A + 0.08 \quad \text{and} \quad B-0.08, \quad c_k \) - varied within the limits of 10 \(^2\)-3.5. Two variants of the mechanical system are considered. In the first variant all elements are viscoelastic (homogeneous system). The results of the calculation are shown in Fig.3. The dependence of both the vibration frequencies \( \omega_{Rj} \) and the damping coefficients (\( j = 1,2 \)) on the parameter \( c_k \) turned out to be monotonic. In the case when all the shock absorbers of the system are elastic, the dependences of the natural frequencies on the parameter coincide with those shown in Fig.4, and with accuracy up to 5%.

Fig. 4. Change the real and imaginary parts of the complex frequencies, depending on the stiffness factor. Dissipative non-homogeneous system.

Dissipative homogeneous mechanical system (all viscoelastic elements with the same rheological properties) is characterized by the fact that in the formula (4), firstly, there is no matrix \( A \) (sub matrices \( A_H \) and \( A_B \) can be transferred to the next matrix) and, secondly, all functions are identical. Then, the system of equations (4) in the form takes the following form:

\[
[f(\omega_R)]A^\alpha - \omega^2 B \overline{\psi} = 0
\]

(5)

where \( A^\alpha \) is the numerical matrix of the total instantaneous rigidities of all viscoelastic elements of the system. After elimination of linearly dependent components from the system (4), the transformed matrices of the generalized instantaneous rigidity \( A^\alpha \) and \( B \) can be written in canonical form, i.e. by a special transformation of generalized coordinates, to bring these matrices to diagonal form. And this means that the mechanical system is a kind of set of independent partial systems with one degree of freedom. In

DOI: http://dx.doi.org/10.24018/ejers.2018.3.6.780
other words, the proper forms of such a system are independent and can be considered and calculated separately from each other. Another position is added for a structurally inhomogeneous viscoelastic system. In (5) we add the matrix of generalized rigidities of the elastic elements of A, and then (5) will look like this:

\[ [A(\omega_R) - \omega^2 B] \ddot{\mathbf{v}} = 0 \]

where \( A(\omega_R) = A + f(\omega_R) \)

In the general case, for two matrices \( A(\omega_R) \) and B (one of which is functional), after preliminary exclusion of linearly dependent components, it is impossible to select simultaneously the no degenerate transformation of the coordinates, leading them to the canonical form. And this means that the proper forms of such a mechanical system cannot be considered separately from each other, i.e. they are addicted. Consequently, with free oscillations between the forms, energy is exchanged. This is especially pronounced if the shapes have close Eigen frequencies.

At the intersection point of the graphs of the damping coefficients \( \omega_{i1} \) and \( \omega_{i2} \), both forms equally scatter energy, although they are different from each other (accurate to the phase). Up to the point \( C = C^* \), energy is "pumped" from the second form to the first, so the latter most intensively dissipates energy. After the intersection point, the difference between the first Eigen frequencies increases, the interaction of the corresponding forms decreases and their dissipative properties takes on a normal character. As an optimization criterion for analyzing the damping properties of a dissipative homogeneous and inhomogeneous mechanical system,

\[ \delta = \omega^0(p_i) = \max_{p_i} \min_{p_j} \omega(p_i) \]

Here \( \omega^0(p_i) \) is the optimal value of the damping index \( \omega^0 \); \( p_i \) (\( p_1, p_2, ..., p_N \)) are optimized parameters (in the problem under consideration, the optimized parameter is one - Z_0); N is the number of natural frequency frequencies being analyzed (\( i = 1, 2, ..., N \)). The practical conclusion is the following: the damping capacity of the design basically determines the minimum damping coefficient \( \omega_{ij} \) in absolute value (in this case, the oscillations of this particular form are damped last) \( \omega_{ij} \). The global (determinant) damping coefficient of the system is first to the point of intersection, and then. The optimum mode of oscillation in the sense of damping will be for \( C = C^* \), when this global damping coefficient is maximal.

IV. FORCED OSCILLATIONS OF MECHANICAL SYSTEMS

In the problem of forced oscillations, generalized forces are assumed to be harmonic in time [9]

\[ \{Q(t)\} = \{Q_0\} e^{-ivt} \]

where \( \{Q_0\} \) and \( V \) are real. Forced oscillations are assumed to be steady, so the lower limit of integration in (2) is assumed to be - \( \infty \). The solution of the problem of forced oscillations is sought in the form.

\[ \{\alpha\} = \{\alpha_0\} e^{-ivt} \]

where the amplitude vector \( \{Q_0\} \) is determined from the system of algebraic equations.

\[ [C]^{-v}[A]\{q_0\} = \{Q_0\} \]

This system was solved in this work by Gauss method. The purpose of this paper is to determine the dependence of the resonant amplitudes \( A_{ij} \) (\( j \)- the number of the generalized coordinate, \( k \)- the number of the resonant frequency) on the parameters of the system.

The system of solids is considered in the quality of the example (Fig. 2). To the body \( M \), the vibrational force \( Q \) (t) is applied in the vertical direction in the form of a periodic function of time. The values \( Q_0 = 1 \) are accepted, the other parameters are the same as those used above. The dependence of the resonant displacements \( A_{ij} \) (\( i, j = 1, 2 \)) on the instantaneous stiffness \( C^* \) is investigated. Two variants of the mechanical system is considered. The results of the calculation for a homogeneous system are shown in Fig. 5. The dependence \( A_{ij} \) on the parameter \( c_k \) turned out to be monotonic (\( i, j = 1,2 \)).

In the second variant, an inhomogeneous system is considered: the first and second deformable elements are elastic \( R_1 = R_2 = 0 \), the remaining parameters coincide with those adopted above.

The results of the count are shown in Figure 6. It is seen that, when the natural frequencies approach each other, the amplitudes \( A_{i1} \) and \( A_{i2} \) become equal. The dependence

![Fig. 5. The change in the resonant amplitude, depending on the stiffness coefficient. Dissipative homogeneous system](image-url)
\[ A_{jk} (j, k = 1, 2) \] on \( c_k \) was not monotonic.

The damping properties of the system as a whole for forced oscillations are determined by the maximum resonance amplitude (we call it the global resonance amplitude). The intensity of dissipative processes in the system is higher, the lower the global resonance amplitude.

\[
\delta_{\lambda} = A^0(p_I) = \min_{p_i} \max_{p_i} [A(p_i)]
\]

In the case of a homogeneous system, the role of the global amplitude is fulfilled for all values of the parameter by the first resonance amplitude. In the case of an inhomogeneous system, both the first and second resonant amplitudes act as the global amplitudes, depending on the parameter value. The roles change, as in the case of the global damping coefficient, with the characteristic value of the parameter at which the real parts of the natural frequencies are closest. At this value of the parameter, the global resonant amplitude is minimal and, consequently, the dissipative processes in the system proceed most intensively. This result is completely consistent with the fact that with the same value of the parameter, the global damping coefficient has a pronounced maximum.

![Fig. 6. The change in the resonant amplitude, depending on the stiffness coefficient. Dissipative non-homogeneous system](image)

**V. CONCLUSIONS**

Thus, the discovered mechanical effect of "Troyanovskii-Safarov" [10]-[13] is confirmed and generalized, which can be formulated as follows: for self-induced and forced oscillations of a structurally inhomogeneous viscoelastic system, its dissipative processes proceed more intensively, the closer the natural frequencies of different normal coordinates. Normal vibrations of an inhomogeneous viscoelastic system with close Eigen frequencies mutually cancel each other.

**REFERENCES**


**Safarov Ismoil Ibrokhimovich** was born on August 10, 1954 in the Bukhara region of the Republic of Uzbekistan. Doctor of Physics and Mathematics, Professor, in the field of mechanics of a deformable solid. In 1972-1977 he graduated from the Tashkent National University, in 1979-1983 and 1989-1992 he studied postgraduate and doctoral studies at the Moscow Institute of Electronic Engineering. He works as the head of the department "Higher Mathematics" of the Tashkent Chemical Technology Institute. His scientific interests are the mechanics of solids and liquids.